STAT 400 Homework 05

Spring 2018 | Dalpiaz | UIUC

Due: *Friday, February 23, 2:00 PM*

Exercise 1

Consider a random variable X with the probability mass function

$$f(x) = \frac{6}{3^x}, \quad x = 2, 3, 4, 5, \ldots$$

(a) Find the moment-generating function of X, $M_X(t)$. Report the function, being sure to indicate the values of t where the function exists.

Solution:

$$\begin{aligned}
M_X(t) = \text{E}\left[e^{tX}\right] &= \sum_x e^{tx} f(x) \\
&= \sum_{x=2}^{\infty} \frac{6e^{tx}}{3^x} \\
&= 6 \sum_{x=2}^{\infty} \left(\frac{e^t}{3}\right)^x \\
&= 6 \cdot \left(\frac{\left(\frac{e^t}{3}\right)^2}{1 - \frac{e^t}{3}}\right) \\
&= \boxed{\frac{2e^{2t}}{3 - e^t}}, \quad \boxed{t < \log(3)}
\end{aligned}$$

The restriction on t is necessary since we require that $\left|\frac{e^t}{3}\right|$ be less than 1, otherwise the sum diverges, and the moment generating function does not exist.

(b) Calculate $\text{E}[X]$.

Solution:

$$M_X'(t) = \frac{(3 - e^t)(4e^{2t}) - (2e^{2t})(-e^t)}{(3 - e^t)^2} = \frac{2e^{2t}(6 - e^t)}{(3 - e^t)^2}$$

$$\text{E}[X] = M_X'(0) = \frac{2(6 - 1)}{(3 - 1)^2} = \boxed{\frac{5}{2}}$$

Alternatively, simply refer to Exercise 3 of Homework 3.

Exercise 2

How much wood would a woodchuck chuck if a woodchuck could chuck wood? Let W denote the amount of wood a woodchuck would chuck per day (in cubic meters) if a woodchuck could chuck wood. Suppose the moment-generating function of W is

$$M_W(t) = 0.1 \cdot e^{3t} + 0.3 \cdot e^{2t} + 0.5 \cdot e^{1t} + 0.1.$$

(a) Calculate the average amount of wood a woodchuck would chuck per day, $\mathrm{E}[W]$.

Solution:

Here, we use the moment generating function to generate the first moment, which is exactly the expected value.

$$M'_W(t) = 0.3 \cdot e^{3t} + 0.6 \cdot e^{2t} + 0.5 \cdot e^{1t}$$

$$\mathrm{E}[W] = M'_W(0) = \boxed{1.4}$$

(b) Calculate $\mathrm{Var}[W]$.

Solution:

Here, we use the moment generating function to generate the second moment.

$$M''_W(t) = 0.9 \cdot e^{3t} + 1.2 \cdot e^{2t} + 0.5 \cdot e^{1t}$$

$$\mathrm{E}[W^2] = M''_W(0) = 2.6$$

We then calculate the variance by using the first and second moments that we had already calcualted.

$$\mathrm{Var}[W] = \mathrm{E}[W^2] - (\mathrm{E}[W])^2 = 2.6 - 1.4^2 = \boxed{0.64}$$

Alternatively, you could have realized that this moment generating function implies thet the probability mass function of W is given by

$$f(w) = \begin{cases} 0.1 & w = 0 \\ 0.5 & w = 1 \\ 0.3 & w = 2 \\ 0.1 & w = 3 \end{cases}$$

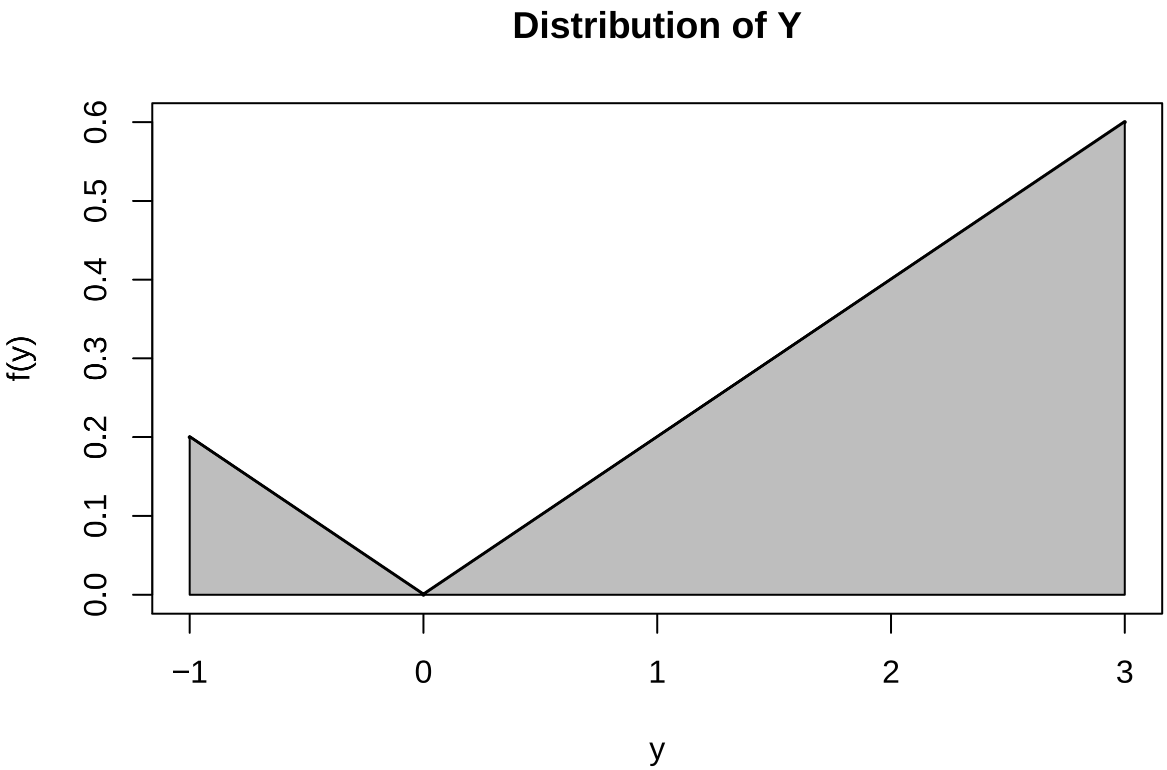
then calculated the expected value and variance using the usual definitions. (Consider doing so for practice if you haven't already.)

Exercise 3

Consider a random variable Y with the probability density function

$$f(y) = \frac{|y|}{5}, \ -1 < y < 3.$$

(a) Calculate $\mathrm{E}[Y]$.



Solution:

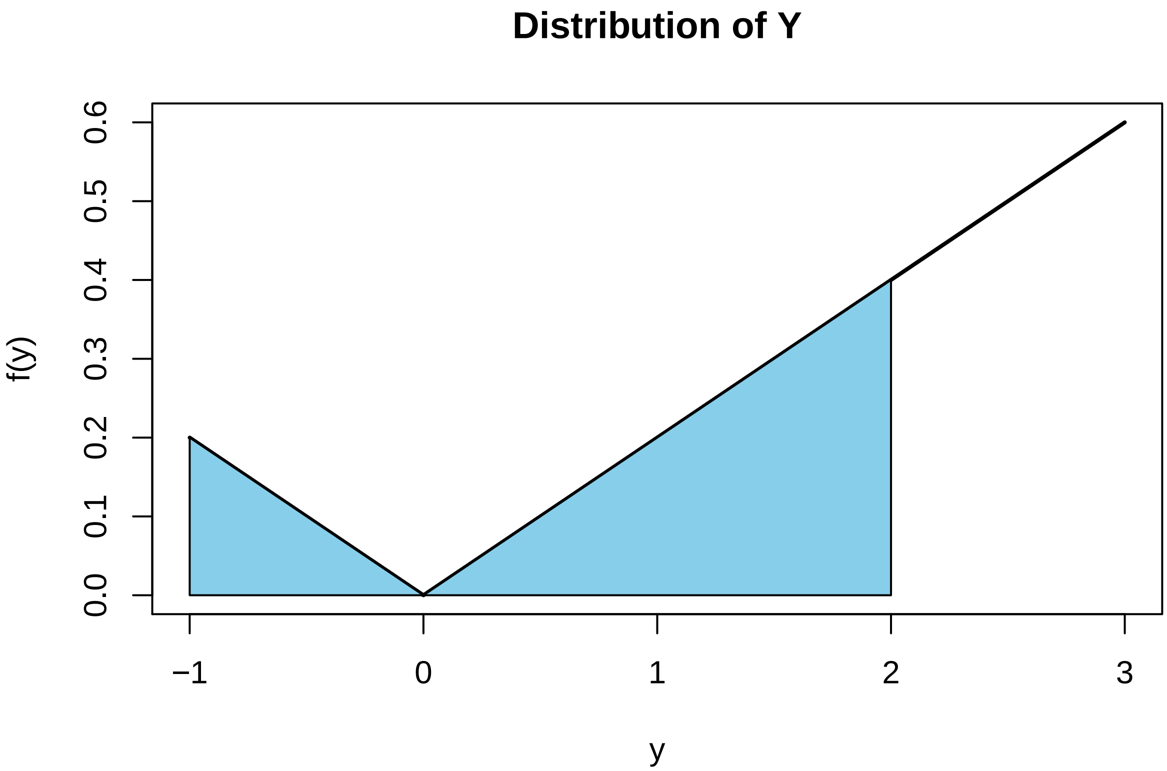
From the picture, it is clear that

$$f(f) = \begin{cases} -0.2y & y < 0 \\ 0.2y & y \geq 0 \end{cases}$$

Then we have

$$\mathrm{E}[Y] = \int_{-\infty}^{\infty} y \cdot f(y) dy = \int_{-1}^{0} y \cdot (-0.2y) dy + \int_{0}^{3} y \cdot (0.2y) dy = -\frac{1}{15} + \frac{9}{5} = \boxed{\frac{26}{15}}$$

(b) Calculate $\mathrm{median}[Y]$, the median of Y.



Solution:

Here we need to find m such that

$$\int_{-\infty}^{m} f(y)dy = 0.5$$

First, note that,

$$\int_{-1}^{0} (-0.2y)dy = 0.1$$

Thus, we know that $m > 0$.

Now, we need

$$\int_{-1}^{0} (-0.2y)dy + \int_{0}^{m} (0.2y)dy = 0.5$$

That is

$$\int_{0}^{m} (0.2y)dy = 0.4$$

Then finally, we have

$$0.1 \cdot m^2 = 0.4$$

This implies that

$$m = \boxed{2}$$

since -2 is outside the possible values of the random variable.

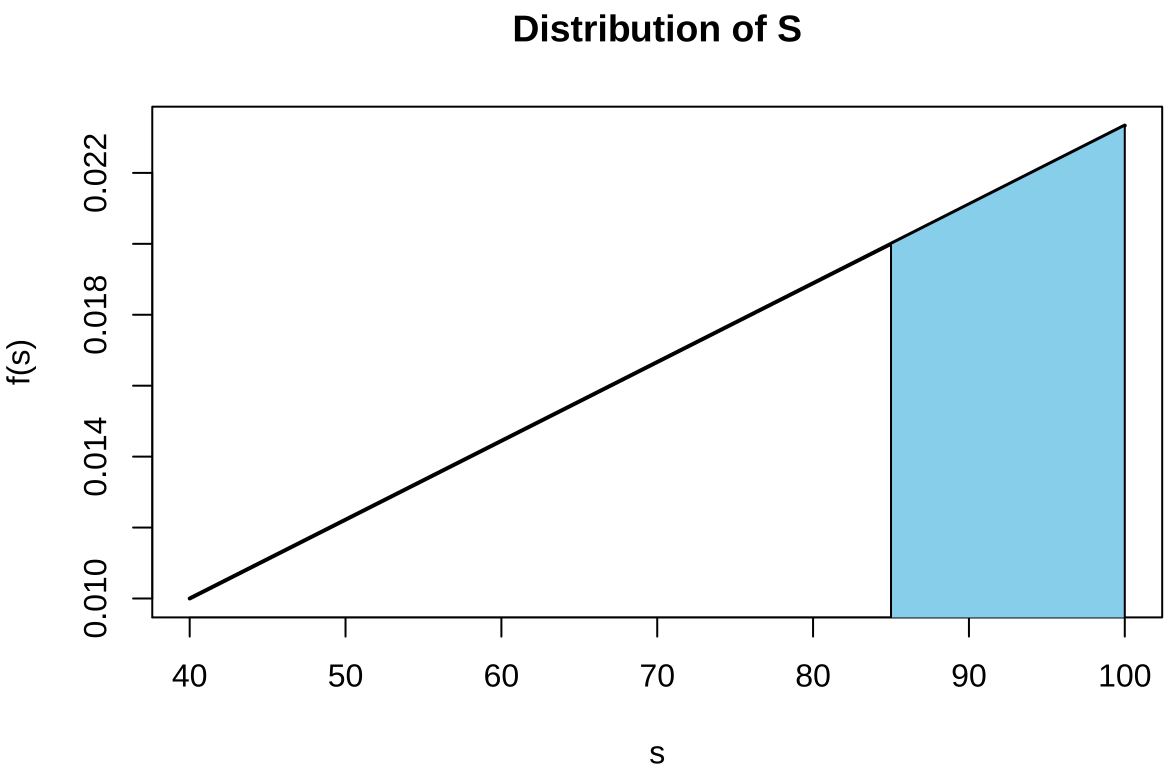
Exercise 4

Suppose that scores on the previous semester's STAT 400 Exam II were not very good. Graphed, their distribution had a shape similar to the probability density function

$$f(s) = \frac{1}{9000}(2s + 10), \quad 40 \leq s \leq 100.$$

Assume that scores on this exam, S, actually follow this distribution. (Note: This distribution does not necessarily reflect reality.)

(a) Suppose 10 students from the class are selected at random. What is the probability that (exactly) 4 of them received a score above 85?



Solution:

Define B to be the number of students how receive an 85 or above. Then $B \sim \text{binom}(N = 10, p)$ where

$$p = \int_{85}^{100} \frac{1}{9000}(2s + 10)ds = \frac{s^2 + 10s}{9000}\Big|_{s=85}^{s=100} = \frac{11000}{9000} - \frac{8075}{9000} = 0.325$$

$$P(B = 4) = \binom{10}{4}(0.325)^4(0.675)^6 \approx \boxed{0.2216}$$

(b) What was the standard deviation of the scores, SD$[S]$?

Solution:

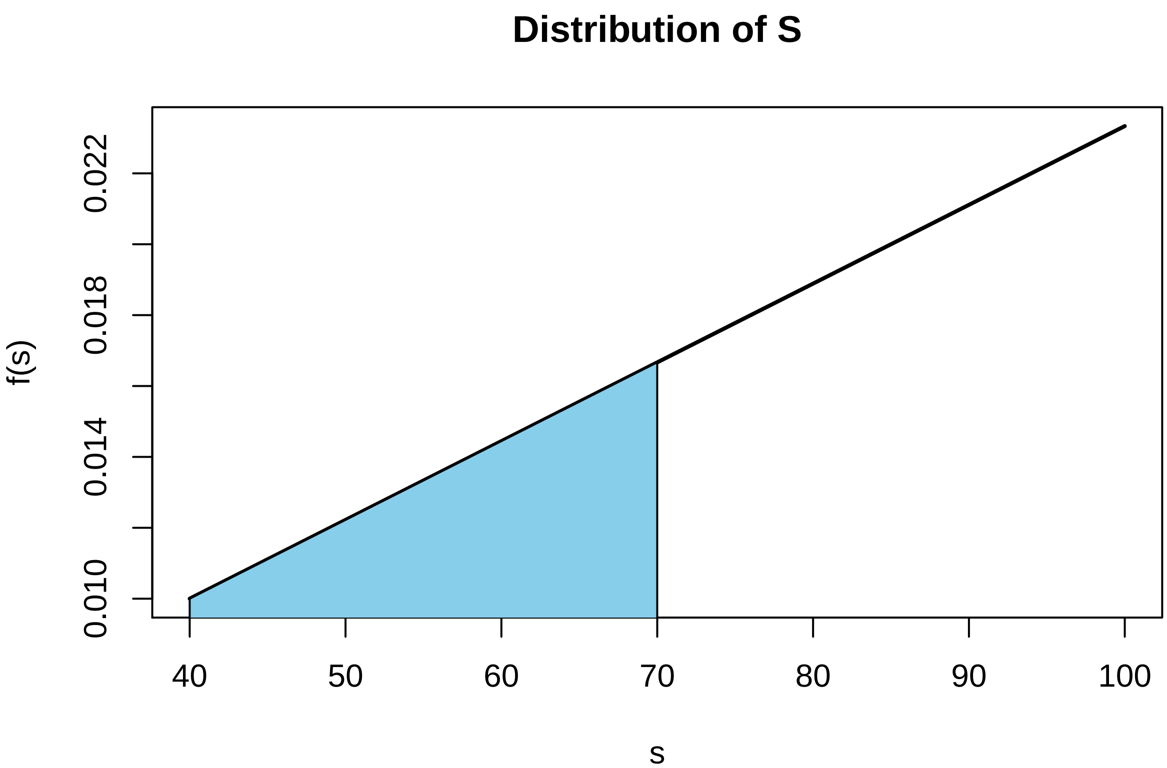
$$\mathrm{E}[S] = \int_{40}^{100} s \cdot \frac{2s+10}{9000}\, ds = 74$$

$$\mathrm{E}[S^2] = \int_{40}^{100} s^2 \cdot \frac{2s+10}{9000}\, ds = 5760$$

$$\mathrm{Var}[S] = \mathrm{E}[S^2] - (\mathrm{E}[S])^2 = 5760 - 74^2 = 284$$

$$\mathrm{SD}[S] = \sqrt{\mathrm{Var}[S]} = \sqrt{284} \approx \boxed{16.8523}$$

(c) What was the class 40th percentile? That is, find a such that $P(S \le a) = 0.40$.



Solution:

Want to find a such that

$$\int_{40}^{a} \frac{2s+10}{9000}\, ds = 0.40$$

That is,

$$\int_{40}^{a} \frac{2s+10}{9000}\, ds = \frac{s^2+10s}{9000}\Big|_{s=40}^{s=a} = \frac{a^2+10a}{9000} - \frac{2}{9} = 0.40$$

Thus,

$$a^2 + 10a - 5600 = 0$$

So -80 and 70 are candidate values.

Finally,

$$a = \boxed{70}$$

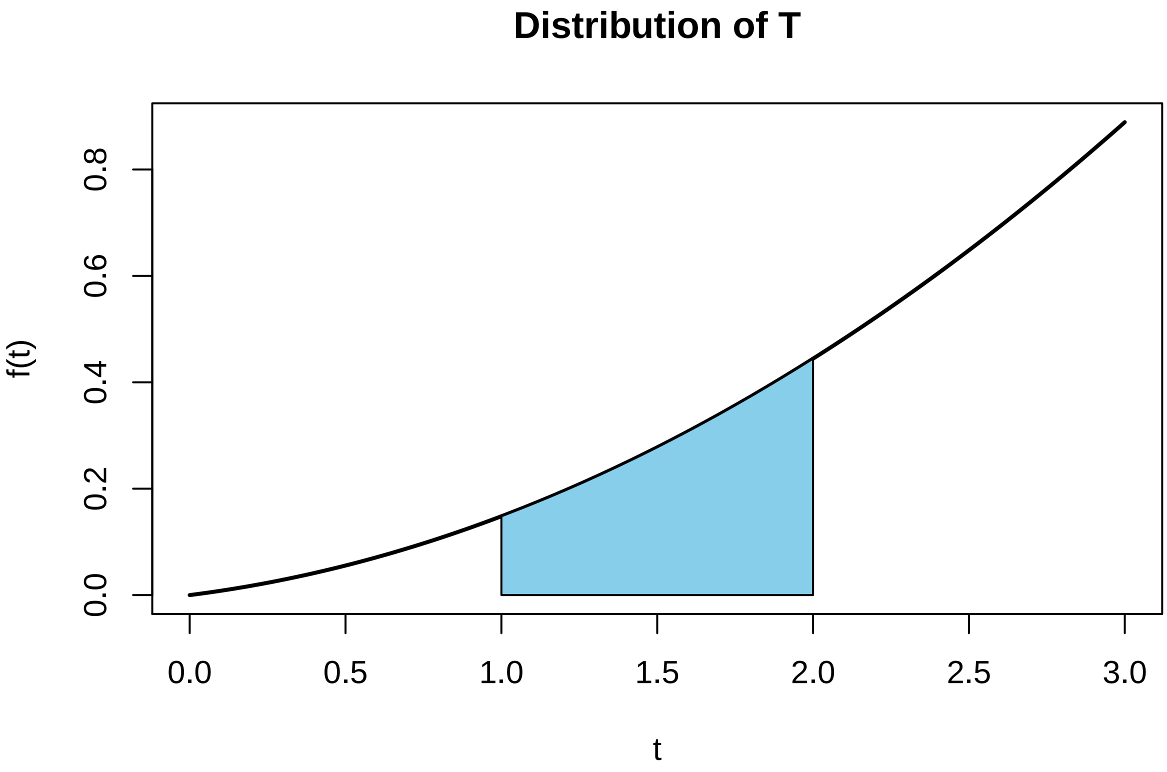
since $40 < 70 < 100$.

Exercise 5

Students often worry about the time it takes to complete an exam. Suppose that completion time in hours, T, for the STAT 400 final exam follows a distribution with density

$$f(t) = \frac{2}{27}(t^2 + t), \quad 0 \le t \le 3.$$

What is the probability that a randomly chosen student finishes the exam during the second hour of the exam. That is, calculate $P(1 < T < 2)$.



Solution:

$$P(1 < T < 2) = \int_1^2 \frac{2}{27}(t^2 + t) = \frac{2}{27}\left(\frac{t^3}{3} + \frac{t^2}{2}\right)\Big|_{t=1}^{t=2} = \frac{28}{81} - \frac{5}{81} = \boxed{\frac{23}{81}}$$