STAT 400 Homework 10

Spring 2018 | Dalpiaz | UIUC

Due: *Friday, April 13, 2:00 PM*

Exercise 1

Before it closed, Ron Swanson was a frequent patron of Charles Mulligan's Steakhouse in Indianapolis, Indiana. Ron enjoyed the experience so much, during each visit he took a picture with his steak.

Ron also weighed each steak he consumed. He has a record of eating six "22 ounce" Charles Mulligan's porterhouse steaks. Ron found that these six steaks weighed

$$22.4 \text{ oz}, \ 20.8 \text{ oz}, \ 21.6 \text{ oz}, \ 20.2 \text{ oz}, \ 21.4 \text{ oz}, \ 22.0 \text{ oz}$$

Suppose that "22 ounce" Charles Mulligan's porterhouse steaks follow a $N(\mu, \sigma^2)$ distribution and that Ron's six steaks were a random sample.

(a) Compute the sample standard deviation, s, of these six steaks. Do not use a computer. You may only use $+$, $-$, $\times$, $\div$, and $\sqrt{}$ on a calculator. Show **all** work.

Solution:

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
22.4	1.0	1.00
20.8	-0.6	0.36
21.6	0.2	0.04
20.2	-1.2	1.44
21.4	0	0.00
22.0	0.6	0.36

So, we have

$$\sum_{i=n}^{n} x_i = 128.4$$

Thus

$$\bar{x} = \frac{1}{n}\sum_{i=n}^{n} x_i = \frac{1}{6}(128.4) = 21.4$$

Note that,

$$\sum_{i=n}^{n}(x_i - \bar{x}) = 0.$$

More importantly,

$$\sum_{i=n}^{n}(x_i - \bar{x})^2 = 3.2$$

Thus,

$$s^2 = \frac{1}{n-1}\sum_{i=n}^{n}(x_i - \bar{x})^2 = \frac{1}{5}(3.2) = 0.64$$

Then finally,

$$s = \sqrt{s^2} = \sqrt{0.64} = \boxed{0.8}$$

```
x = c(22.4, 20.8, 21.6, 20.2, 21.4, 22.0)
mean(x)
```

```
## [1] 21.4
```

```
var(x)
```

```
## [1] 0.64
```

```
sd(x)
```

```
## [1] 0.8
```

(b) Construct a 95% two-sided confidence interval for the true mean weight of a "22 ounce" Charles Mulligan's porterhouse steak, μ.

Solution:

Here σ is unknown and n is small, so we use t.

$$\bar{x} \pm t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}$$

We have,

- $n = 6$,
- $\bar{x} = 21.4$,

- $s = 0.8$,
- $1 - \alpha = 0.95$, so $\alpha/2 = 0.025$,
- $t_{\alpha/2}(n-1) = t_{0.025}(5) = 2.571$.

$$21.4 \pm 2.571 \frac{0.8}{\sqrt{6}}$$

$$\boxed{\mathbf{21.4 \pm 0.8397}}$$

$$\boxed{(\mathbf{20.5603, 22.2397})}$$

```
n = length(x)
est = mean(x)
s = sd(x)
alpha = 0.05
crit = qt(alpha / 2, df = n - 1, lower.tail = FALSE)
se = s / sqrt(n)
margin = crit * se
lower = est - margin
upper = est + margin
c(lower, upper) # above answer contains some rounding, hence the difference
```

```
## [1] 20.56045 22.23955
```

```
t.test(x, level = 0.95)$conf.int
```

```
## [1] 20.56045 22.23955
## attr(,"conf.level")
## [1] 0.95
```

(c) Construct a 95% confidence lower bound for the true mean weight of a "22 ounce" Charles Mulligan's porterhouse steak, μ.

Solution:

$$\left[\bar{x} - t_{\alpha}(n-1)\frac{s}{\sqrt{n}}, \infty\right)$$

We have,

- $n = 6$,
- $\bar{x} = 21.4$,
- $s = 0.8$,
- $1 - \alpha = 0.95$, so $\alpha = 0.05$,
- $t_{\alpha}(n-1) = t_{0.05}(5) = 2.0150$.

$$21.4 - 2.0150 \frac{0.8}{\sqrt{6}} = 20.7419$$

$$\boxed{[20.7419, \infty)}$$

```
n = length(x)
est = mean(x)
s = sd(x)
alpha = 0.05
crit = qt(alpha, df = n - 1, lower.tail = FALSE)
se = s / sqrt(n)
margin = crit * se
(lower = est - margin)
```

```
## [1] 20.74189
```

(d) Construct a 90% two-sided confidence interval for the true standard deviation of the weight of a "22 ounce" Charles Mulligan's porterhouse steak, σ.

Solution:

$$\left(\sqrt{\frac{(n-1) \cdot s^2}{\chi^2_{\alpha/2}(n-1)}}, \sqrt{\frac{(n-1) \cdot s^2}{\chi^2_{1-\alpha/2}(n-1)}} \right)$$

- $n = 6$,
- $s = 0.8$,
- $1 - \alpha = 0.90$, so $\alpha/2 = 0.05$, and $1 - \alpha/2 = 0.95$
- $\chi^2_{\alpha/2}(n-1) = \chi^2_{0.05}(5) = 11.07$.
- $\chi^2_{1-\alpha/2}(n-1) = \chi^2_{0.95}(5) = 1.145$.

$$\left(\sqrt{\frac{(6-1) \cdot 0.8^2}{11.07}}, \sqrt{\frac{(6-1) \cdot 0.8^2}{1.145}} \right)$$

$$\boxed{(0.5377, 1.6718)}$$

```
n = 6
s = 0.8
alpha = 0.10
crit_lower = qchisq(alpha / 2, df = n - 1, lower.tail = FALSE)
crit_upper = qchisq(1 - alpha / 2, df = n - 1, lower.tail = FALSE)
lower = sqrt((n - 1) * s ^ 2 / crit_lower)
upper = sqrt((n - 1) * s ^ 2 / crit_upper)
c(lower, upper)
```

```
## [1] 0.5376398 1.6714060
```

(e) Construct a 90% confidence upper bound for the true standard deviation of the weight of a "22 ounce" Charles Mulligan's porterhouse steak, σ.

Solution:

$$\left(0, \sqrt{\frac{(n-1) \cdot s^2}{\chi^2_{1-\alpha}(n-1)}} \right)$$

- $n = 6$,
- $s = 0.04$,
- $1 - \alpha = 0.90$
- $\chi^2_{1-\alpha}(n-1) = \chi^2_{0.90}(5) = 1.610$.

$$\left(0, \sqrt{\frac{(6-1)\cdot 0.8^2}{1.610}}\right)$$

$$\boxed{(0, 1.4098)}$$

```
n = 6
s = 0.8
alpha = 0.10
crit_upper = qchisq(1 - alpha, df = n - 1, lower.tail = FALSE)
upper = sqrt((n - 1) * s ^ 2 / crit_upper)
c(0, upper)
```

```
## [1] 0.00000 1.40968
```

Exercise 2

Last year, ballots in Champaign-Urbana contained the following question to assess public opinion on an issue:

> "Should the State of Illinois legalize and regulate the sale and use of marijuana in a similar fashion as the State of Colorado?"

Suppose that we would like to understand Champaign-Urbana's 2017 opinion on marijuana legalization. To satisfy our curiosity, we obtain a random sample of 120 Champaign-Urbanians and find that 87 support marijuana legalization.

(a) Construct a 99% confidence interval for p, the true proportion of Champaign-Urbanians that support marijuana legalization.

Solution:

- $x = 87$
- $n = 120$
- $\alpha = 0.01$, $\alpha/2 = 0.005$
- $z_{\alpha/2} = z_{0.005} = 2.576$

$$\hat{p} = \frac{x}{n} = \frac{87}{120} = 0.725$$

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}\cdot(1-\hat{p})}{n}}$$

$$0.725 \pm 2.576\sqrt{\frac{0.725\cdot 0.275}{120}}$$

$$\boxed{\mathbf{0.725 \pm 0.1050}}$$

$$\boxed{(\mathbf{0.62}, \mathbf{0.83})}$$

(b) Suppose that a pollster wants to estimate the true proportion of Champaign-Urbanians that support marijuana legalization to within 0.04, with 95% confidence. How many Champaign-Urbanians should this pollster poll? Assume the pollster has no prior knowledge about the proportion.

Solution:

- $\hat{p} = 0.50$, since we have no prior knowledge. (Worst case scenario.)
- $\epsilon = 0.04$
- $\alpha = 0.05$, $\alpha/2 = 0.025$
- $z_{\alpha/2} = z_{0.025} = 1.960$

$$n = \left\lceil \left(\frac{z_{\alpha/2}}{\epsilon}\right)^2 \cdot \hat{p}(1-\hat{p}) \right\rceil = \left\lceil \left(\frac{1.960}{0.04}\right)^2 \cdot 0.50 \cdot 0.50 \right\rceil = \boxed{601}$$

(c) Now assume the pollster believes that support for legalization is somewhere between 65% and 85% and they would like to estimate the true proportion of Champaign-Urbanians that support marijuana legalization to within 0.04, with 90% confidence. How many Champaign-Urbanians should this pollster poll?

Solution:

- $\tilde{p} = 0.65$, since it is closest to 0.50. (Worst case scenario.)
- $\epsilon = 0.04$
- $\alpha = 0.10$, $\alpha/2 = 0.05$
- $z_{\alpha/2} = z_{0.05} = 1.645$

$$n = \left\lceil \left(\frac{z_{\alpha/2}}{\epsilon}\right)^2 \cdot \tilde{p}(1-\tilde{p}) \right\rceil = \left\lceil \left(\frac{1.645}{0.04}\right)^2 \cdot 0.65 \cdot 0.35 \right\rceil = \boxed{385}$$

Exercise 3

Suppose students in a Statistics class are interested in the average score of an exam, but the instructor has only graded (a random sample of) 13 of the (many) exams. The instructor states that a 90% confidence interval for the true mean is given by $(79.14, 82.86)$ and that you can assume the grades follow a normal distribution.

Using only this information, calculate $\bar{x}$, s, and finally, a 95% confidence interval for μ, the true mean of the exam.

Solution:

First, since the interval is symmetric about the sample mean

$$\bar{x} = \frac{79.14 + 82.86}{2} = 81$$

We can also obtain the margin of error, which is half the length of the interval,

$$\epsilon = \frac{82.86 - 79.14}{2} = 1.86$$

The critical value of an 90% confidence interval with 13 observations is

$$t_{0.05}(12) = 1.782$$

Then solving

$$1.86 = 1.782 \cdot \frac{s}{\sqrt{13}}$$

we find that

$$s = 3.7634.$$

The critical value of an 95% confidence interval with 13 observations is

$$t_{0.025}(12) = 2.1788$$

$$\bar{x} \pm t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}$$

Then a 95% confidence interval is given by

$$81 \pm 2.1788 \frac{3.7634}{\sqrt{13}}$$

$$\boxed{\mathbf{81 \pm 2.2742}}$$

$$\boxed{\mathbf{(78.7258, 83.2742)}}$$

Exercise 4

Suppose that 10 students visit the Stars Hollow Apple Orchard and each pick (a random sample of) 15 Fuji apples, weigh them, then create a 90% confidence interval for the true mean weight of a Fuji apple at the Stars Hollow Apple Orchard. What is the probably that at most 2 of these intervals do not contain the true mean weight of a Fuji apple at the Stars Hollow Apple Orchard?

Since we haven't actually seen any of the intervals, and assuming they are created with the correct procedure, each has a 90% of containing the true mean, thus they have a 10% chance to not contain the true mean.

Define X to be the number of intervals that do not contain the true mean. Then,

$$X \sim \text{binom}(n = 10, p = 0.10)$$

So finally we want

$$\begin{aligned} P(X \le 2) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &= 0.3486784 + 0.3874205 + 0.1937102 \\ &= \boxed{0.9298092} \end{aligned}$$

```
dbinom(c(0, 1, 2), size = 10, prob = 0.10)
```

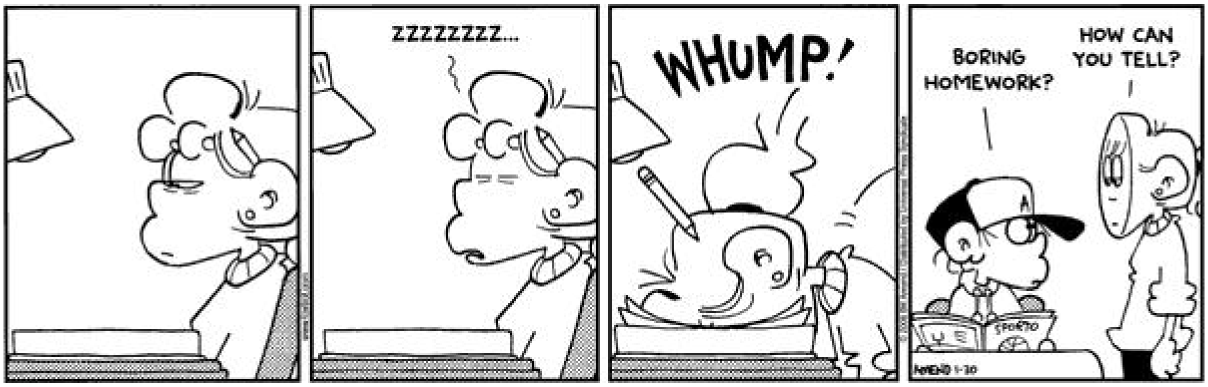
```
## [1] 0.3486784 0.3874205 0.1937102
```

```
sum(dbinom(c(0, 1, 2), size = 10, prob = 0.10))
```

```
## [1] 0.9298092
```

```
pbinom(2, size = 10, prob = 0.10)
```

```
## [1] 0.9298092
```


ZZZZZZZZ...
WHUMP!
BORING HOMEWORK?
HOW CAN YOU TELL?
SPORTS