STAT 400 Spring 2018

Discussion 01 Solutions

1. Consider the following experiment:

A letter is chosen at random from the word **S T A T I S T I C S**.

a) List all possible outcomes and their probabilities.

10 letters: 1 **A**, 1 **C**, 2 **I**, 3 **S**, 3 **T**.

Possible Outcomes:	**A**,	**C**,	**I**,	**S**,	**T**
Probabilities:	0.10	0.10	0.20	0.30	0.30

b) What is the probability that the letter selected is a vowel?

P(vowel) = P(**A**) + P(**I**) = 0.10 + 0.20 = **0.30**.

2. **1.1-4** **1.2-4**

A fair coin is tossed four times, and the sequence of heads and tails is observed.

a) List each of the 16 sequences in the sample space *S*.

H H H H	H T H H	T H H H	T T H H
H H H T	H T H T	T H H T	T T H T
H H T H	H T T H	T H T H	T T T H
H H T T	H T T T	T H T T	T T T T

b) Let events A, B, C, and D be given by A = { at least 3 heads }, B = { at most 2 heads }, C = { heads on the third toss }, and D = { 1 head and 3 tails }. If the probability set function assigns 1/16 to each outcome in the sample space, find

(i) $P(A)$, (ii) $P(A \cap B)$, (iii) $P(B)$,

(iv) $P(A \cap C)$, (v) $P(D)$, (vi) $P(A \cup C)$

(vii) $P(B \cap D)$.

A	B	C	D
H H H H	H H T T	H H H H	H T T T
H H H T	H T H T	H H H T	T H T T
H H T H	H T T H	H T H H	T T H T
H T H H	H T T T	H T H T	T T T H
T H H H	T H H T	T H H H	
	T H T H	T H H T	
	T H T T	T T H H	
	T T H H	T T H T	
	T T H T		
	T T T H		
	T T T T		

(i) $P(A) = \mathbf{5/16}$

(ii) $A \cap B = \varnothing$ $P(A \cap B) = \mathbf{0}$

(iii) $P(B) = \mathbf{11/16}$

(iv) $A \cap C = \{ HHHH, HHHT, HTHH, THHH \}$
$P(A \cap C) = \mathbf{4/16}$

(v) $P(D) = \mathbf{4/16}$

(vi) $P(A \cup C) = P(A) + P(C) - P(A \cap C) = 5/16 + 8/16 - 4/16 = \mathbf{9/16}$

(vii) $D \subset B$ $B \cap D = D$ $P(B \cap D) = P(D) = \mathbf{4/16}$

3. **1.1-6** **1.2-8**

If $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cap B) = 0.3$, find …

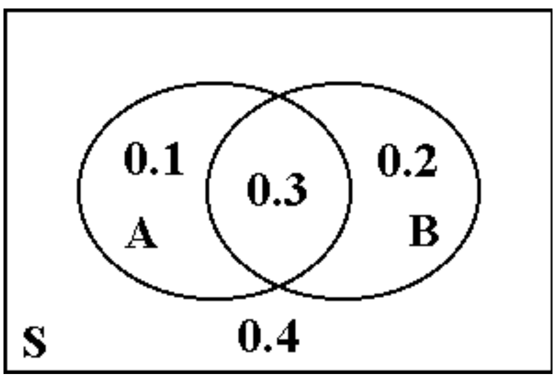
a) $P(A \cup B)$; b) $P(A \cap B')$; c) $P(A' \cup B')$.

a) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.5 - 0.3 = \mathbf{0.6}$.

b) $P(A \cap B') = P(A) - P(A \cap B) = 0.4 - 0.3 = \mathbf{0.1}$.

c) $P(A' \cup B') = 1 - P(A \cap B) = 1 - 0.3 = \mathbf{0.7}$.

OR



a) $P(A \cup B) = \mathbf{0.6}$.

b) $P(A \cap B') = \mathbf{0.1}$.

c) $P(A' \cup B') = \mathbf{0.7}$.

4. Suppose that $P(A) = 0.40$, $P(B) = 0.50$, $P(A \cup B) = 0.70$. Find …

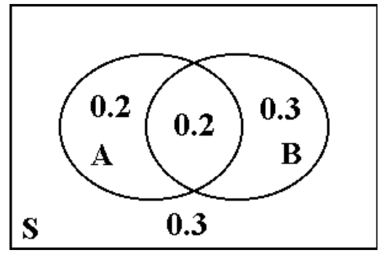
a) $P(A \cap B)$; b) $P(A' \cap B')$; c) $P(A' \cup B')$.

a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.7 = 0.4 + 0.5 - P(A \cap B)$ $P(A \cap B) = \mathbf{0.2}$.

b) $P(A' \cap B') = 1 - P(A \cup B) = 1 - 0.7 = \mathbf{0.3}$.

c) $P(A' \cup B') = 1 - P(A \cap B) = 1 - 0.2 = \mathbf{0.8}$.



5. **1.2-16**

The five numbers 1, 2, 3, 4, and 5 are written respectively on five disks of the same size and placed in a hat. Two disks are drawn without replacement from the hat, and the numbers written on them are observed.

a) List the 10 possible outcomes of this experiment as unordered pairs of numbers.

$(1, 2)$	$(1, 3)$	$(1, 4)$	$(1, 5)$	$(2, 3)$
$(2, 4)$	$(2, 5)$	$(3, 4)$	$(3, 5)$	$(4, 5)$

b) If each of the 10 outcomes has probability 1/10, assign a value to the probability that the sum of the two numbers drawn is

(i) 3; (ii) between 6 and 8 inclusive.

(i) $\mathbf{1/10}$ (ii) $\mathbf{5/10}$

6. Suppose that $P(A) = 0.60$ and $P(B) = 0.50$.

a) Can A and B be mutually exclusive? Why or why not? What is the minimum possible value of $P(A \cap B)$? What is the maximum possible value of $P(A \cap B)$?

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

If A and B are mutually exclusive, $P(A \cap B) = 0$.

No, A and B cannot be mutually exclusive since $P(A) + P(B) = 1.1 > 1$.

$[\text{Minimum possible value of } P(A \cap B)] = \mathbf{0.10}$.

$[\text{Maximum possible value of } P(A \cap B)] = \mathbf{0.50}$ (the smaller of P(A) and P(B)).

b) What is the minimum possible value of $P(A \cup B)$? What is the maximum possible value of $P(A \cup B)$?

$[\text{Minimum possible value of } P(A \cup B)] = \mathbf{0.60}$ (the larger of P(A) and P(B)).

$[\text{Maximum possible value of } P(A \cup B)] = \mathbf{1}$.

7. Suppose that $P(A) = 0.40$ and $P(B) = 0.30$.

a) Can A and B be mutually exclusive? Why or why not? What is the minimum possible value of $P(A \cap B)$? What is the maximum possible value of $P(A \cap B)$?

Yes, A and B can be mutually exclusive since $P(A) + P(B) = 0.7 \le 1$.

$[\text{Minimum possible value of } P(A \cap B)] = \mathbf{0}$.

$[\text{Maximum possible value of } P(A \cap B)] = \mathbf{0.30}$ (the smaller of P(A) and P(B)).

b) What is the minimum possible value of $P(A \cup B)$? What is the maximum possible value of $P(A \cup B)$?

$[\text{Minimum possible value of } P(A \cup B)] = \mathbf{0.40}$ (the larger of P(A) and P(B)).

$[\text{Maximum possible value of } P(A \cup B)] = \mathbf{0.70}$.

8. Suppose

$P(A) = 0.40$,

$P(B) = 0.34$,

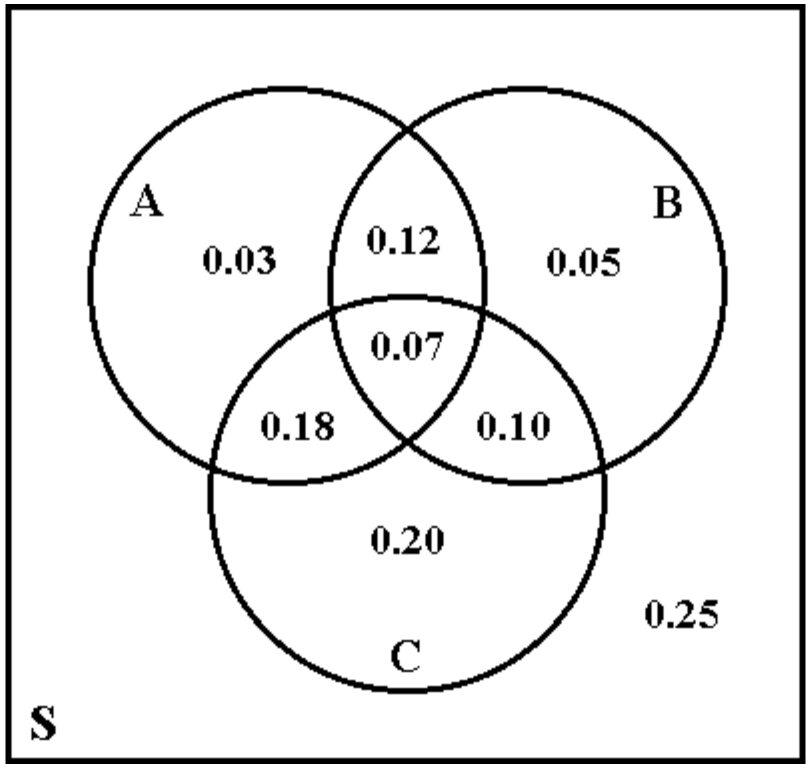
$P(C) = 0.55$,

$P(A \cap B) = 0.19$,

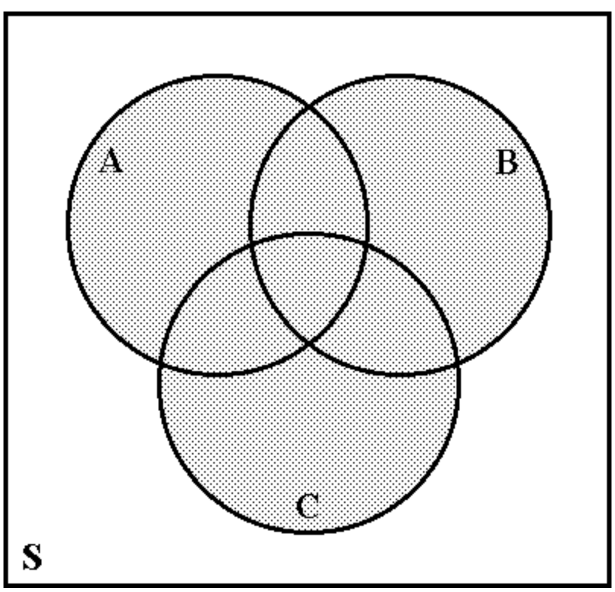
$P(A \cap C) = 0.25$,

$P(B \cap C) = 0.17$,

$P(A \cap B \cap C) = 0.07$.

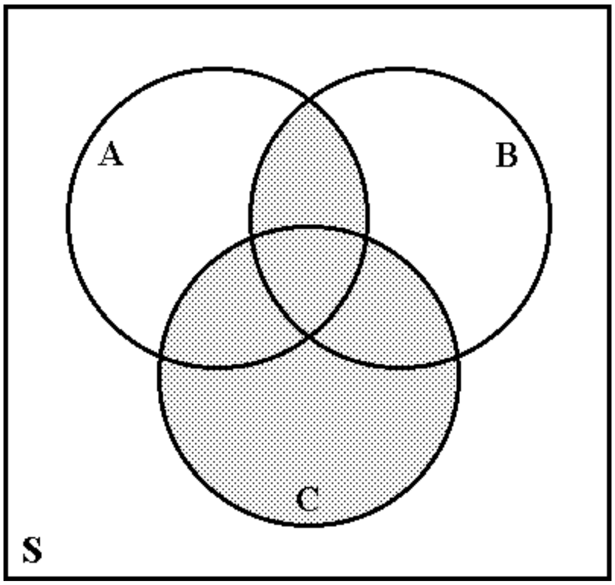


a) Find $P(A \cup B \cup C)$.



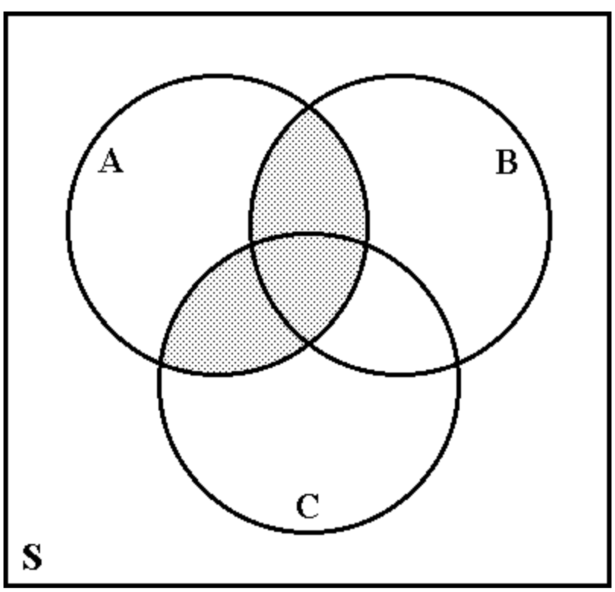
$P(A \cup B \cup C) = \mathbf{0.75}$.

b) Find $P((A \cap B) \cup C)$.



$P((A \cap B) \cup C) = \mathbf{0.67}$.

c) Find $P(A \cap (B \cup C))$.



$P(A \cap (B \cup C)) = \mathbf{0.37}$.

9. Find the value of p that would make this a valid probability model.

a) Suppose S = { 0, 2, 4, 6, 8, … } (even non-negative integers) and

$$P(0) = p, \qquad P(k) = \frac{1}{3^k}, \quad k = 2, 4, 6, 8, \ldots .$$

$$1 = p(0) + p(2) + p(4) + p(6) + p(8) + \ldots$$

$$= p + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \frac{1}{3^8} + \ldots = p + \frac{\frac{1}{9}}{1 - \frac{1}{9}} = p + \frac{1}{8}. \qquad \Rightarrow \qquad p = \mathbf{\frac{7}{8}}.$$

b) Suppose S = { 1, 2, 3, 4, … } (positive integers) and

$$P(1) = p, \qquad P(k) = \frac{(\ln 3)^k}{k!}, \quad k = 2, 3, 4, \ldots .$$

Must have $\sum_{\text{all } x} p(x) = 1.$ $\qquad \Rightarrow \qquad p(1) + \sum_{k=2}^{\infty} \frac{(\ln 3)^k}{k!} = 1.$

$$\sum_{k=2}^{\infty} \frac{(\ln 3)^k}{k!} = \sum_{k=0}^{\infty} \frac{(\ln 3)^k}{k!} - 1 - \ln 3 = e^{\ln 3} - 1 - \ln 3 = 2 - \ln 3.$$

$p(1) + 2 - \ln 3 = 1.$ $\qquad \Rightarrow \qquad p(1) = \mathbf{\ln 3 - 1} \approx 0.0986.$

10. If $P(A) = 0.7$, $P(B) = 0.5$, and $P(A' \cap B') = 0.1$, find …

a) $P(A \cap B)$; b) $P(A \mid B)$; c) $P(B \mid A)$.

a) $P(A \cup B) = 1 - P(A' \cap B') = 1 - 0.1 = 0.9;$

$$0.9 = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

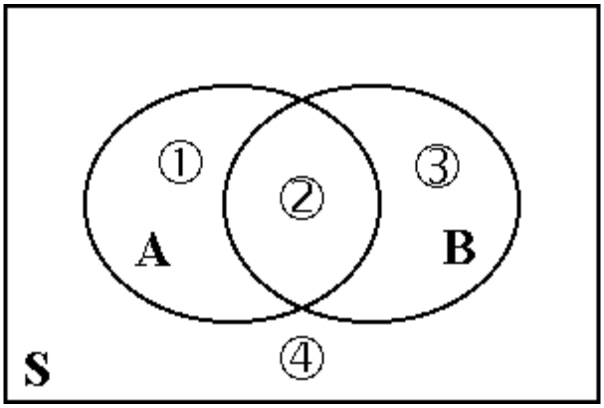
$$= 0.7 + 0.5 - P(A \cap B),$$

$$P(A \cap B) = \mathbf{0.3}.$$

b) $$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.5} = \frac{3}{5} = \mathbf{0.6}.$$

c) $$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.3}{0.7} = \mathbf{\frac{3}{7}}.$$

11. Suppose $P(B) = 0.45$, $P(A \cup B) = 0.67$, $P(A \cup B') = 0.89$.



$P(B) = ② + ③ = 0.45,$

$P(A \cup B) = ① + ② + ③ = 0.67,$

$P(A \cup B') = ① + ② + ④ = 0.89.$

$① + ② + ③ + ④ = 1.$

$\Rightarrow$ ① $= 0.22,$ ③ $= 0.11,$ ④ $= 0.33.$

$\Rightarrow$ ② $= 0.34.$

a) Find $P(A)$.

$P(A) = ① + ② = \mathbf{0.56}.$

b) Find $P(A \mid B')$.

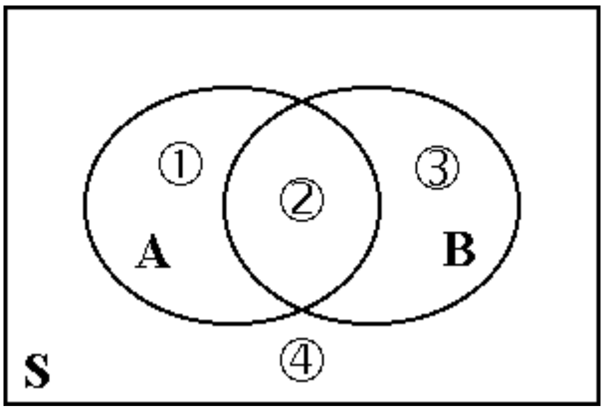
$$P(A \mid B') = \frac{P(A \cap B')}{P(B')} = \frac{0.22}{0.55} = \mathbf{\frac{2}{5}} = \mathbf{0.40}.$$

c) Find $P(B' \mid A')$.

$$P(B' \mid A') = \frac{P(A' \cap B')}{P(A')} = \frac{0.33}{0.44} = \mathbf{\frac{3}{4}} = \mathbf{0.75}.$$

12. a) 1.1-7 1.2-9

Given that $P(A \cup B) = 0.76$ and $P(A \cup B') = 0.87$, find $P(A)$.



$P(A \cup B) = ① + ② + ③ = 0.76,$

$P(A \cup B') = ① + ② + ④ = 0.87.$

$① + ② + ③ + ④ = 1.$

$\Rightarrow \quad ③ = 0.13, \quad ④ = 0.24.$

$\Rightarrow \quad P(A) = ① + ② = \mathbf{0.63}.$

b) Suppose also that $P(B) = 0.20$. Find …

i) $P(A \cap B)$; ii) $P(A \mid B)$; iii) $P(B \mid A)$.

i) $P(B) = ② + ③ = 0.20,$ $③ = 0.13.$

$\Rightarrow \quad P(A \cap B) = ② = \mathbf{0.07}.$

ii) $$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.07}{0.20} = \mathbf{0.35}.$$

iii) $$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.07}{0.63} = \mathbf{\frac{1}{9}}.$$

13. Suppose an individual is randomly selected from the population of adult males living in the United States. Let A be the event that the selected individual is over 6 ft. in height, and let B be the event that the selected individual is a professional basketball player. Which do you think is larger, $P(A|B)$ or $P(B|A)$, and **why**? (*Circle one and **explain**.*)

$P(A|B) < P(B|A)$ $\qquad$ $P(A|B) \approx P(B|A)$ $\qquad$ $\boxed{P(A|B) > P(B|A)}$

$P(A|B)$ = proportion of professional basketball players that are over 6 ft. in height. Since almost all professional basketball players are over 6 ft. in height (with a few exceptions), $P(A|B)$ is close to 1.0.

$P(B|A)$ = proportion of adult males over 6 ft. in height that are professional basketball players. A fairly small proportion adult males over 6 ft. in height are professional basketball players.

$\mathbf{P(A|B)}$ is the larger of the two.

OR

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \qquad P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

Since $P(A) > P(B)$ (there are a lot more adult males who are over 6 ft. in height than professional basketball players), $\mathbf{P(A|B)}$ is the larger of the two.

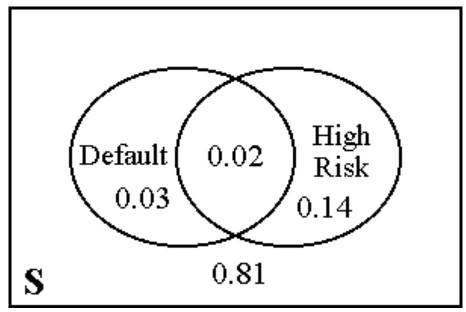
14. A bank classifies borrowers as "high risk" or "low risk," and 16% of its loans are made to those in the "high risk" category. Of all the bank's loans, 5% are in default. It is also known that 40% of the loans in default are to high-risk borrowers.

P(High risk) = 0.16, P(Default) = 0.05, P(High risk | Default) = 0.40.

a) What is the probability that a randomly selected loan is in default and issued to a high-risk borrower?

$$P(\text{Default} \cap \text{High risk}) = P(\text{Default}) \cdot P(\text{High risk} \mid \text{Default}) = 0.05 \cdot 0.40 = \mathbf{0.02}.$$

	High Risk	Low Risk	
Default	**0.02**	0.03	**0.05**
Default′	0.14	0.81	0.95
	0.16	0.84	1.00



b) What is the probability that a loan will default, given that it is issued to a high-risk borrower?

$$P(\text{Default} \mid \text{High risk}) = \frac{P(\text{Default} \cap \text{High risk})}{P(\text{High risk})} = \frac{0.02}{0.16} = \mathbf{0.125}.$$

c) What is the probability that a randomly selected loan is either in default or issued to a high-risk borrower, or both?

$$P(\text{Default} \cup \text{High risk}) = P(\text{Default}) + P(\text{High risk}) - P(\text{Default} \cap \text{High risk})$$

$$= 0.05 + 0.16 - 0.02 = \mathbf{0.19}.$$

d) A loan is being issued to a borrower who is not high-risk. What is the probability that this loan will default?

$$P(\text{Default} \mid \text{High risk}') = \frac{P(\text{Default} \cap \text{High risk}')}{P(\text{High risk}')} = \frac{0.03}{0.84} \approx \mathbf{0.0357}.$$

15. At *Initech*, 60% of all employees surf the Internet during work hours. 24% of the employees surf the Internet and play MMORPG during work hours. It is also known that 72% of the employees either surf the Internet or play MMORPG (or both) during work hours.

$$P(\text{ Internet }) = 0.60, \qquad P(\text{ Internet} \cap \text{MMORPG }) = 0.24,$$

$$P(\text{ Internet} \cup \text{MMORPG }) = 0.72.$$

a) What proportion of the employees play MMORPG during work hours?

$$P(\text{ Internet} \cup \text{MMORPG }) = P(\text{ Internet }) + P(\text{ MMORPG }) - P(\text{ Internet} \cap \text{MMORPG })$$

$$0.72 = 0.60 + P(\text{ MMORPG }) - 0.24$$

$$P(\text{ MMORPG }) = \mathbf{0.36}.$$

	MMORPG	MMORPG′	
Internet	**0.24**	0.36	**0.60**
Internet′	0.12	0.28	0.40
	0.36	0.64	1.00

b) If it is known that an employee surfs the Internet during work hours, what is the probability that he/she also plays MMORPG?

$$P(\text{ MMORPG} \mid \text{Internet }) = \frac{P(\text{MMORPG} \cap \text{Internet})}{P(\text{Internet})} = \frac{0.24}{0.60} = \mathbf{0.40}.$$

c) Suppose an employee does not play MMORPG during work hours. What is the probability that he/she surfs the Internet?

$$P(\text{ Internet} \mid \text{MMORPG}') = \frac{P(\text{Internet} \cap \text{MMORPG}')}{P(\text{MMORPG}')} = \frac{0.36}{0.64} = \frac{\mathbf{9}}{\mathbf{16}} = \mathbf{0.5625}.$$

16. Suppose $S = \{1, 2, 3, \ldots\}$ and $P(x) = \left(\frac{1}{2}\right)^x, \quad x = 1, 2, 3, \ldots$.

a) Find P(outcome is divisible by 3).

P(outcome is divisible by 3) = P(3) + P(6) + P(9) + P(12) + …

$$= \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \frac{1}{2^{12}} + \ldots = \sum_{k=1}^{\infty} \left(\frac{1}{8}\right)^k = \frac{1}{8} \cdot \frac{1}{1 - 1/8} = \mathbf{\frac{1}{7}}.$$

b) Find P(outcome is divisible by 3 | outcome is divisible by 2).

P(outcome is divisible by 2) = P(2) + P(4) + P(6) + P(8) + …

$$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + \ldots = \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k = \frac{1}{4} \cdot \frac{1}{1 - 1/4} = \frac{1}{3}.$$

P(outcome is divisible by 6) = P(6) + P(12) + P(18) + P(24) + …

$$= \frac{1}{2^6} + \frac{1}{2^{12}} + \frac{1}{2^{18}} + \frac{1}{2^{24}} + \ldots = \sum_{k=1}^{\infty} \left(\frac{1}{64}\right)^k = \frac{1}{64} \cdot \frac{1}{1 - 1/64} = \frac{1}{63}.$$

P(outcome is divisible by 3 | outcome is divisible by 2)

$$= \frac{P(\text{outcome divisible by } 3 \cap \text{outcome divisible by } 2)}{P(\text{outcome divisible by } 2)}$$

$$= \frac{P(\text{outcome divisible by } 6)}{P(\text{outcome divisible by } 2)} = \frac{1/63}{1/3} = \mathbf{\frac{1}{21}}.$$

17. 500 people, all of whom drive approximately 10,000 miles per year, were classified according to age and the number of auto accidents each has had during the last three years:

Number of Accidents	*Age (in years)*		
	Under 40	**Over 40**	Total
0	170	80	250
1	80	70	150
More than 1	50	50	100
Total	300	200	500

A person is selected at random from those 500.

a) What is the probability that the person selected is over 40 and has had more than 1accident?

$^{50}/_{500}$ = **0.10**.

b) What is the probability that the person selected is either over 40 or has had more than 1 accident (or both)?

$^{250}/_{500}$ = **0.50**.

c) Find the probability that the person selected is over 40 given that he/she has had more than 1 accident.

$^{50}/_{100}$ = **0.50**.

d) Suppose that the person selected is over 40. What is the probability that he/she has had more than 1 accident?

$^{50}/_{200}$ = **0.25**.

e) Find the probability that the person selected is over 40 given that he/she has had at most 1 accident.

$^{150}/_{400}$ = **0.375**.

f) Find the probability that the person selected has had more than 1 accident given that he/she has had at least one accident.

$^{100}/_{250}$ = **0.40**.

18. **1.3-4 (c)** **1.4-4 (c)**

Two cards are drawn successively and without replacement from an ordinary deck of playing cards. Compute the probability of drawing a heart on the first draw and an ace on the second draw.

Hint: Note that a heart can be drawn by getting the ace of hearts or one of the other 12 hearts.

$$P(\text{Non-Ace Heart, Ace}) + P(\text{Ace of Hearts, Non-Heart Ace})$$

$$= \frac{12}{52}\cdot\frac{4}{51} + \frac{1}{52}\cdot\frac{3}{51} = \frac{48+3}{52\cdot 51} = \frac{51}{52\cdot 51} = \mathbf{\frac{1}{52}}.$$

19. **1.3-7** **1.4-9**

An urn contains four colored balls: two orange and two blue. Two balls are selected at random without replacement, and you are told that at least one of them is orange. What is the probability that the other ball is also orange?

Hint 1: O O   O B   B O   B B

Hint 2: $P(\text{both orange} \mid \text{at least one orange})$.

Outcome	O O	O B	B O	B B
Probability	$\frac{2}{4}\times\frac{1}{3} = \frac{2}{12}$	$\frac{2}{4}\times\frac{2}{3} = \frac{4}{12}$	$\frac{2}{4}\times\frac{2}{3} = \frac{4}{12}$	$\frac{2}{4}\times\frac{1}{3} = \frac{2}{12}$

$$P(\text{both orange} \mid \text{at least one orange}) = \frac{P(\text{both orange} \cap \text{at least one orange})}{P(\text{at least one orange})}$$

$$= \frac{P(\{OO\} \cap \{OO, OB, BO\})}{P(\{OO, OB, BO\})} = \frac{P(\{OO\})}{P(\{OO, OB, BO\})} = \frac{2/12}{10/12} = \mathbf{\frac{1}{5}}.$$

20. Bowl A contains three red and two white chips, and bowl B contains four red and three white chips. A chip is drawn at random from bowl A and transferred to bowl B.

a) **1.3-16** **1.4-18**

Compute the probability of then drawing a red chip from bowl B.

$$P(R_B) = P(R_A R_B) + P(W_A R_B)$$

$$= P(R_A) \times P(R_B \mid R_A) + P(W_A) \times P(R_B \mid W_A)$$

$$= \frac{3}{5} \cdot \frac{5}{8} + \frac{2}{5} \cdot \frac{4}{8} = \frac{15}{40} + \frac{8}{40} = \mathbf{\frac{23}{40}} = \mathbf{0.575}.$$

b) <u>If a red chip was drawn from bowl B</u>, what is the probability that a red chip had been drawn from bowl A?

$$P(R_A \mid R_B) = \frac{P(R_A R_B)}{P(R_B)} = \frac{\frac{15}{40}}{\frac{23}{40}} = \mathbf{\frac{15}{23}} \approx 0.652.$$