STAT 400 | Spring 2018

Discussion 06 Solutions

1. The weight of fish in Lake Paradise follows a normal distribution with mean of 8.1 lbs and standard deviation of 2.5 lbs.

a) What proportion of fish are between 9 lbs and 12 lbs?

$$P(9 < X < 12) = P\left(\frac{9-8.1}{2.5} < Z < \frac{12-8.1}{2.5}\right) = P(0.36 < X < 1.56)$$

$$= 0.9406 - 0.6406 = \mathbf{0.3}.$$

b) Alex boasts that he once caught a fish that was just big enough to be in the top 2.5% of of the fish population. How much did his fish weigh?

Want $x = ?$ such that $P(X > x) = 0.025$.

$P(Z > 1.96) = 0.0250.$ $\quad x = 8.1 + 2.5 \times 1.96 = \mathbf{13}$ lbs.

c) If one catches a fish from the bottom 20% of the population, the fish must be returned to the lake. What is the weight of the smallest fish that one can keep?

Want $x = ?$ such that $P(X < x) = 0.20$.

$P(Z < -0.84) = 0.2005 \approx 0.20.$ $\quad x = 8.1 + 2.5 \times (-0.84) = \mathbf{6}$ lbs.

2. Suppose the duration of trouble-free operation of a particular brand of refrigerators is normally distributed with mean 105 months and standard deviation 12 months.

a) If the company that makes this particular brand of refrigerators wishes to set the warranty period so that only 4% of all refrigerators would need repair services while under warranty, how long a warranty must be set?

Need $x = ?$ such that $P(X < x) = 0.04$.

① Find z such that $P(Z < z) = 0.04$.

The area to the left is **0.04** $= \Phi(z)$.

Using the standard normal table, $z = \mathbf{-1.75}$.

② $x = \mu + \sigma \cdot z$. $x = 105 + 12 \cdot (-1.75) =$ **84 months (7 years)**.

b) What is the probability that a refrigerator of this brand will work for over 8 years without trouble?

8 years = 96 months.

$$P(X > 96) = P\left(Z > \frac{96 - 105}{12}\right) = P(Z > -0.75) = \mathbf{0.7734}.$$

c) Sixteen refrigerators of this brand are purchased for a 16-unit apartment complex. What is the probability that exactly 10 out of 16 refrigerators would work for over 8 years without trouble? (Assume independence.)

Let W denote the number of refrigerators (out of 16) that work for over 8 years without trouble.

Then W has **Binomial** Distribution, $n = 16$, $p = \mathbf{0.7734}$ (see part (b)).

$$P(W = 10) = {}_{16}C_{10} \cdot (0.7734)^{10} \cdot (1 - 0.7734)^{6} \approx \mathbf{0.083}.$$

3. Bob sells thingamabobs. His yearly salary is \$27,000 plus a commission of 10% of his sales. His yearly sales are normally distributed with mean \$100,000 and standard deviation \$20,000.

a) Find the probability that Bob earns over \$40,000 in a given year.

Let X denote Bob's yearly sales. Let Y denote the amount Bob earns in a given year.

Then $\quad Y = 0.10 \times X + 27{,}000.$

Since X has a Normal distribution with mean \$100,000 and standard deviation \$20,000, Y also has a Normal distribution with mean $0.10 \times 100{,}000 + 27{,}000 = \$37{,}000$ and standard deviation $|0.10| \times 20{,}000 = \$2{,}000$.

$$P(Y > 40{,}000) = P\left(Z > \frac{40{,}000 - 37{,}000}{2{,}000}\right) = P(Z > 1.50) = \mathbf{0.0668}.$$

b) Find the missing value: With probability 67% Bob earns over _______ in a given year.

Need $y = ?$ such that $P(Y > y) = 0.67$.

① Find z such that $P(Z > z) = 0.67$.

The area to the left is $\mathbf{0.33} = \Phi(z)$.

Using the standard normal table, $\quad z = -\mathbf{0.44}$.

② $x = \mu + \sigma \cdot z. \qquad x = 37{,}000 + 2{,}000 \cdot (-0.44) = \mathbf{\$36{,}120}$.

4. **3.3-6** **3.6-6**

If the moment-generating function of X is $M(t) = \exp(166\,t + 200\,t^2)$, find

a) The mean of X.

b) The variance of X.

c) $P(170 < X < 200)$.

d) $P(148 < X < 172)$.

$M(t) = e^{166t+400t^2/2}$ so

(a) $\mu = 166$; **(b)** $\sigma^2 = 400$;

(c) $P(170 < X < 200) = P(0.2 < Z < 1.7) = 0.3761$;

(d) $P(148 \le X \le 172) = P(-0.9 \le Z \le 0.3) = 0.4338$.

5. Suppose X is a normally distributed random variable. Suppose also that

$$P(X > 111) = 0.8413 \quad \text{and} \quad P(X < 177) = 0.9599.$$

Find the mean and the standard deviation of X.

$P(X > 111) = 0.8413$

$P(Z > -1.00) = 0.8413$

$\mu - 1.00\,\sigma = 111$

$P(X < 177) = 0.9599$.

$P(Z < 1.75) = 0.9599$.

$\mu + 1.75\,\sigma = 177$

$\mu - 1.00\,\sigma = 111$

$\mu + 1.75\,\sigma = 177$

$\Rightarrow$ $2.75\,\sigma = 66$ $\Rightarrow$ $\sigma =$ **24**.

$\Rightarrow$ $\mu =$ **135**.

6.* Let the random variable X have the p.d.f.

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}, \qquad 0 < x < \infty, \qquad \text{zero elsewhere.}$$

Find the mean and the variance of X.

Hint: Compute $\mathrm{E}(\mathrm{X})$ directly (after an obvious substitution).
Find $\mathrm{E}(\mathrm{X}^2)$ by comparing the integral with the integral representing $\mathrm{Var}(\mathrm{Z}) = \mathrm{E}(\mathrm{Z}^2)$ of a random variable Z that is $\mathrm{N}(0, 1)$.

$$\mathrm{E}(\mathrm{X}) = \int_0^\infty x \frac{2}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \ldots$$

$$u = \frac{x^2}{2} \qquad du = x\, dx$$

$$\ldots = \int_0^\infty \frac{2}{\sqrt{2\pi}} e^{-u}\, du = \frac{2}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}.$$

$$\mathrm{E}(\mathrm{X}^2) = \int_0^\infty x^2 \frac{2}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \int_{-\infty}^\infty x^2 \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \mathrm{E}(\mathrm{Z}^2) = 1.$$

$$(\mathrm{Z} \sim \mathrm{N}(0, 1))$$

OR

$$\mathrm{E}(\mathrm{X}^2) = \int_0^\infty x^2 \frac{2}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \ldots \qquad u = \frac{x^2}{2} \qquad du = x\, dx$$

$$\ldots = \int_0^\infty \frac{2}{\sqrt{2\pi}} \sqrt{2u}\, e^{-u}\, du = \frac{2}{\sqrt{\pi}} \int_0^\infty u^{1/2} e^{-u}\, du = \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = 1.$$

$$\mathrm{Var}(\mathrm{X}) = \mathrm{E}(\mathrm{X}^2) - [\mathrm{E}(\mathrm{X})]^2 = 1 - \frac{2}{\pi} = \frac{\pi - 2}{\pi}.$$

7. A computer independently generates seven random numbers from a $\text{Uniform}(0, 1)$ distribution.

a) What is the probability that exactly three will be in the interval from ½ to 1?

Let X denote the number of random numbers (out of 7) that are between ½ and 1.

Then X has a Binomial distribution, $n = 7$, $p = 0.50$.

$P(X = 3) = {}_7C_3 \times 0.50^3 \times 0.50^4 = \mathbf{0.2734375}.$

b) What is the probability that fewer than three will be in the interval from ¾ to 1?

Let X denote the number of random numbers (out of 7) that are between ¾ and 1.

Then X has a Binomial distribution, $n = 7$, $p = 0.25$.

$$P(X < 3) = {}_7C_0 \times 0.25^0 \times 0.75^7 + {}_7C_1 \times 0.25^1 \times 0.75^6 + {}_7C_2 \times 0.25^2 \times 0.75^5$$

$$\approx 0.1334839 + 0.3114624 + 0.3114624 = \mathbf{0.7564087}.$$

8. Alex sets three alarm clocks each night to ensure that he does not sleep through his 9:00 a.m. class. His primary clock works properly on 90% of the mornings, his second alarm clock works properly on 80% of mornings, and his third alarm clock is an antique, it works properly on only 60% of mornings. Assume the alarm clocks are independent.

a) Find the probability that Alex's three-alarm strategy prevents him from oversleeping. That is, find the probability that at least one alarm clock would work on a given morning.

$1 - P(1' \cap 2' \cap 3') = 1 - 0.10 \times 0.20 \times 0.40 = 1 - 0.008 = \mathbf{0.992}.$

b) Find the probability that all three alarm clocks would work on a given morning.

$P(1 \cap 2 \cap 3) = 0.90 \times 0.80 \times 0.60 = \mathbf{0.432}.$

c) Find the probability that exactly two alarm clocks would work on a given morning.

1	2	3′	$0.90 \times 0.80 \times 0.40 = 0.288$
1	2′	3	$0.90 \times 0.20 \times 0.60 = 0.108$
1′	2	3	$0.10 \times 0.80 \times 0.60 = 0.048$

$0.288 + 0.108 + 0.048 = \mathbf{0.444}.$

9. Alex owns a dog and a cat. He is a neglectful pet owner. On any given day, Alex remembers to feed the dog with probability 0.75. The cat is less lucky, the probability that Alex remembers to feed the cat is only 0.60. There is also a 20% chance that neither animal gets fed.

$P(D) = 0.75, \qquad P(C) = 0.60, \qquad P(D' \cap C') = 0.20.$

a) What is the probability that Alex remembers to feed both the dog and the cat?

$P(D \cup C) = 1 - P(D' \cap C') = 1 - 0.20 = 0.80.$

$P(D \cup C) = P(D) + P(C) - P(D \cap C).$

$0.80 = 0.75 + 0.60 - P(D \cap C).$ $\qquad P(D \cap C) = \mathbf{0.55}.$

OR

$P(D' \cup C') = P(D') + P(C') - P(D' \cap C') = 0.25 + 0.40 - 0.20 = 0.45.$

$P(D \cap C) = 1 - P(D' \cup C') = 1 - 0.45 = \mathbf{0.55}.$

b) Suppose Alex forgets to feed the dog. What is the probability that Alex remembers to feed the cat?

	C	C′	
D	0.55	0.20	0.75
D′	0.05	0.20	0.25
	0.60	0.40	1.00

$P(C \mid D') = \frac{0.05}{0.25} = \mathbf{\frac{1}{5}} = \mathbf{0.20}.$

c) Are events { Alex remembers to feed the dog } and { Alex remembers to feed the cat } independent? *Justify your answer.*

$0.55 = P(D \cap C) \neq P(D) \cdot P(C) = 0.75 \cdot 0.60 = 0.45.$

D and C are **NOT independent**.

OR

$0.20 = P(D' \cap C') \neq P(D') \cdot P(C') = 0.25 \cdot 0.40 = 0.10.$

D' and C' are NOT independent.

$\Rightarrow$ D and C are **NOT independent**.

OR

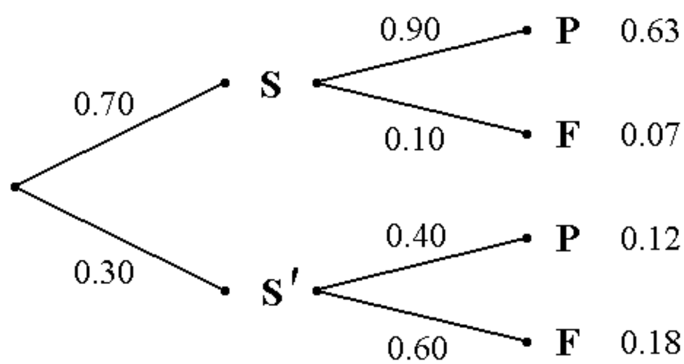
$0.20 = P(C \mid D') \neq P(C) = 0.60.$

D' and C are NOT independent.

$\Rightarrow$ D and C are **NOT independent**.

10. Alex is a naughty student – he does not always study for his exams. There is only a 70% chance that he would study for an exam. If he does study for an exam, the probability that he would pass it is 0.90. However, if he does not study, there is an 60% chance he would fail.

$P(S) = 0.70,$ $P(P \mid S) = 0.90,$ $P(F \mid S') = P(P' \mid S') = 0.60.$



OR

	P	$F = P'$	Total
S	$0.70 \cdot \mathbf{0.90}$ 0.63	0.07	**0.70**
S'	0.12	$0.30 \cdot \mathbf{0.60}$ 0.18	0.30
Total	0.75	0.25	1.00

a) Suppose you find out that Alex passed an exam. What is the probability that he did not study for it?

$$P(S' \mid P) = \frac{P(S' \cap P)}{P(P)} = \frac{0.12}{0.63 + 0.12} = \frac{0.12}{0.75} = \mathbf{0.16}.$$

OR

Bayes' Theorem:

$$P(S' \mid P) = \frac{P(S') \cdot P(P \mid S')}{P(S) \cdot P(P \mid S) + P(S') \cdot P(P \mid S')} = \frac{0.30 \cdot 0.40}{0.70 \cdot 0.90 + 0.30 \cdot 0.40} = \frac{0.12}{0.75} = \mathbf{0.16}.$$

b) Suppose you find out that Alex failed an exam. What is the probability that he did study for it?

$$P(S \mid F) = \frac{P(S \cap F)}{P(F)} = \frac{0.07}{0.07 + 0.18} = \frac{0.07}{0.25} = \mathbf{0.28}.$$

OR

Bayes' Theorem:

$$P(S \mid F) = \frac{P(S) \cdot P(F|S)}{P(S) \cdot P(F|S) + P(S') \cdot P(F|S')} = \frac{0.70 \cdot 0.10}{0.70 \cdot 0.10 + 0.30 \cdot 0.60} = \frac{0.07}{0.25} = \mathbf{0.28}.$$

11. Alex is a naughty student – he does not always do all the problems on a homework assignment. Suppose a certain homework assignment has 18 problems, but Alex only did 7 of them. The grader will randomly select 5 of the 18 problems to grade. What is the probability that Alex did at least 3 of the 5 problems chosen for grading?

18		or	18		or	18	
↙	↘		↙	↘		↙	↘
7 done	11 not done		7 done	11 not done		7 done	11 not done
↓	↓		↓	↓		↓	↓
3	2		4	1		5	0

$$\frac{{}_7C_3 \cdot {}_{11}C_2}{{}_{18}C_5} + \frac{{}_7C_4 \cdot {}_{11}C_1}{{}_{18}C_5} + \frac{{}_7C_5 \cdot {}_{11}C_0}{{}_{18}C_5} = \frac{35 \cdot 55}{8568} + \frac{35 \cdot 11}{8568} + \frac{21 \cdot 1}{8568}$$

$$= \frac{2331}{8568} = \mathbf{\frac{37}{136}} \approx \mathbf{0.272}.$$

12. Alex uses a copy machine to make 75 copies of the exam. Suppose that the stapler independently malfunctions with probability 3% for each copy of the exam.

a) Find the probability that the stapler would malfunction exactly 4 times.

Let X = number of times the stapler malfunctions.

Binomial distribution, $n = 75$, $p = 0.03$.

$$P(X = 4) = \binom{75}{4} 0.03^{4}\, 0.97^{71} \approx \mathbf{0.1132}.$$

b) Use Poisson approximation to find the probability that the stapler would malfunction exactly 4 times.

Poisson approximation: $\lambda = n \cdot p = 75 \cdot 0.03 = 2.25.$

$$P(X = 4) = \frac{2.25^{4} \cdot e^{-2.25}}{4!} \approx \mathbf{0.1126}.$$

13. Suppose that 70%of all cars made by a certain manufacturer meet the new EPA standards. If a random sample of 10 cars is taken, what is the probability that at least 8 of them meet the new EPA standards?

Let X = number of cars (out of 7) that meet the new EPA standards.

Then X has Binomial distribution, $n = 7$, $p = 0.80$.

$$P(X \geq 8) = {}_{10}C_8 \cdot (0.70)^8 \cdot (0.30)^2 + {}_{10}C_9 \cdot (0.70)^9 \cdot (0.30)^1 + {}_{10}C_{10} \cdot (0.70)^{10} \cdot (0.30)^0$$

$$\approx 0.2335 + 0.1211 + 0.0282 = \mathbf{0.3828}.$$

15. Suppose that the number of CFUs (colony forming units) of E. coli in ground beef has Poisson distribution, with an average of 0.8 CFUs per pound of ground beef. What is the probability that there are exactly 3 CFUs in a four-pound package of ground beef?

4 pounds $\Rightarrow$ $\lambda = 0.8 \times 4 = 3.2$.

$$P(X = 3) = \frac{3.2^3 \cdot e^{-3.2}}{3!} = \mathbf{0.2226}.$$

15. The pumpkin diameters at Peter Peter Pumpkin Eater Pumpkin Patch are normally distributed with mean 16.5 inches and standard deviation 2 inches.

a) What proportions of pumpkins have diameter over 15 inches?

$$P(X > 15) = P\left(Z > \frac{15 - 16.5}{2}\right) = P(Z > -0.75) = \mathbf{0.7734}.$$

b) Pumpkins with diameters in the top 3% are set aside for a pumpkin-carving contest. Find the minimum diameter a pumpkin must have to be set aside for the contest.

Want $P(X > x) = 0.03$.

① Find z such that $P(Z > z) = 0.03$. $z = 1.88$.

② $x = \mu + \sigma \cdot z = 16.5 + 2 \cdot 1.88 = \mathbf{20.26}$ inches.

c) If 8 pumpkins are selected at random, what is the probability that exactly 5 of them have diameters over 15 inches?

Let Y = number of pumpkins (out of 8) that have diameters over 15 inches.

Then Y has Binomial distribution, $n = 8$, $p = 0.7734$ (see part (a)).

$$P(Y = 5) = {}_8C_5 \cdot (0.7734)^5 \cdot (0.2266)^3 \approx \mathbf{0.1803}.$$

16. Suppose a discrete random variable X has the following probability mass function:

$$f(0)=p, \qquad f(k)=\frac{1}{6^k}, \qquad k=1,2,3,4,5,6,7,\ldots.$$

a) Find the value of p that makes this is a valid probability distribution.

Must have $\sum_{\text{all } x} f(x)=1.$

$$1 = p+\sum_{k=1}^{\infty}\frac{1}{6^k} = p+\frac{\frac{1}{6}}{1-\frac{1}{6}} = p+\frac{1}{5}. \qquad \Rightarrow \qquad f(0)=p=\mathbf{\frac{4}{5}}.$$

b) Find the moment-generating function of X, $M_X(t)$. For which values of t does it exist?

$$M_X(t) = E(e^{tX}) = e^{0t}\cdot\frac{4}{5}+\sum_{k=1}^{\infty}e^{tk}\cdot\frac{1}{6^k} = \frac{4}{5}+\sum_{k=1}^{\infty}\left(\frac{e^t}{6}\right)^k = \frac{4}{5}+\frac{\frac{e^t}{6}}{1-\frac{e^t}{6}}.$$

$$= \frac{4}{5}+\frac{e^t}{6-e^t} = \frac{4}{5}+\frac{6}{6-e^t}-1 = \frac{6}{6-e^t}-\frac{1}{5}.$$

Need $\left|\frac{e^t}{6}\right|<1. \qquad \Rightarrow \qquad t<\ln 6\approx 1.79176.$

$$M_X(t) = \frac{4}{5}+\frac{e^t}{6-e^t} = \frac{6}{6-e^t}-\frac{1}{5}, \qquad t<\ln 6.$$

c) Find $E(X)$.

$$M_X'(t)=\frac{d}{dt}\left(\frac{4}{5}+\frac{e^t}{6-e^t}\right)=\frac{e^t\left(6-e^t\right)-e^t\left(-e^t\right)}{\left(6-e^t\right)^2}=\frac{6\cdot e^t}{\left(6-e^t\right)^2}.$$

OR

$$M'_X(t) = \frac{d}{dt}\left(\frac{6}{6-e^t} - \frac{1}{5}\right) = -\frac{6}{\left(6-e^t\right)^2}\cdot\left(-e^t\right) = \frac{6\cdot e^t}{\left(6-e^t\right)^2}.$$

$$E(X) = M'_X(0) = \frac{6}{(6-1)^2} = \mathbf{\frac{6}{25}} = \mathbf{0.24}.$$

OR

$$E(X) = 0\cdot\frac{4}{5} + 1\cdot\frac{1}{6^1} + 2\cdot\frac{1}{6^2} + 3\cdot\frac{1}{6^3} + 4\cdot\frac{1}{6^4} + 5\cdot\frac{1}{6^5} + 6\cdot\frac{1}{6^6} + \ldots$$

$$\frac{1}{6}\cdot E(X) = \qquad 1\cdot\frac{1}{6^2} + 2\cdot\frac{1}{6^3} + 3\cdot\frac{1}{6^4} + 4\cdot\frac{1}{6^5} + 5\cdot\frac{1}{6^6} + \ldots$$

$$\Rightarrow \quad \left(1-\frac{1}{6}\right)\cdot E(X) = \frac{1}{6^1} + \frac{1}{6^2} + \frac{1}{6^3} + \frac{1}{6^4} + \frac{1}{6^5} + \frac{1}{6^6} + \ldots = \sum_{k=1}^{\infty}\frac{1}{6^k} = \frac{\frac{1}{6}}{1-\frac{1}{6}} = \frac{1}{5}.$$

$$\Rightarrow \quad \frac{5}{6}\cdot E(X) = \frac{1}{5}.$$ Therefore, $E(X) = \mathbf{\frac{6}{25}} = \mathbf{0.24}$.

OR

$$E(X) = 0\cdot\frac{4}{5} + \sum_{k=1}^{\infty} k\cdot\frac{1}{6^k} = \frac{1}{5}\cdot\left[\sum_{k=1}^{\infty} k\cdot\left(\frac{1}{6}\right)^{k-1}\cdot\frac{5}{6}\right] = \frac{1}{5}\cdot E(Y),$$

where Y is a Geometric random variable with probability of “success” $\frac{5}{6}$.

$$E(Y) = \frac{1}{p} = \frac{6}{5}.$$ Therefore, $E(X) = \mathbf{\frac{6}{25}} = \mathbf{0.24}$.

17. Suppose that number of accidents at a construction site follows a Poisson process with the average rate of 0.80 accidents per month. Assume all months are independent of each other.

a) Find the probability that exactly 2 accidents will occur in one month.

1 month $\Rightarrow \lambda = 0.80.$ $\qquad$ $P(X = 2) = \dfrac{0.80^{2} \cdot e^{-0.80}}{2!} = \mathbf{0.1438}.$

b) Find the probability that at least 2 accidents will occur in one month.

1 month $\Rightarrow \lambda = 0.80.$

$$P(X \ge 2) = 1 - [P(X = 0) + P(X = 1)] = 1 - \left[\frac{0.80^{0} \cdot e^{-0.80}}{0!} + \frac{0.80^{1} \cdot e^{-0.80}}{1!}\right]$$

$$= 1 - [0.4493 + 0.3595] = \mathbf{0.1912}.$$

c) Find the probability that 7 accidents will occur in one year (12 months).

12 months $\Rightarrow \lambda = 0.80 \cdot 12 = 9.6.$ $\qquad$ $P(X = 7) = \dfrac{9.6^{7} \cdot e^{-9.6}}{7!} = \mathbf{0.10098}.$

d) Find the probability that there will be 4 accident-free months in one year.

Let Y = the number of accident-free months in one year.

Then Y has Binomial distribution, $n = 12$, $p = 0.4493$ $\qquad$ (Poisson, $\lambda = 0.80$)

$$P(Y = 4) = {}_{12}C_{4} \cdot (0.4493)^{4} \cdot (1 - 0.4493)^{8} = \mathbf{0.1706}.$$

18. If $E(X) = 75$ and $E(X^2) = 5949$, use Chebyshev's inequality to determine

$\mu = E(X) = 75$, $\sigma^2 = \text{Var}(X) = 5949 - 75^2 = 324.$

$\sigma = \text{SD}(X) = 18.$

a) A lower bound for $P(0 < X < 150)$.

By Chebyshev's Inequality, for $\varepsilon > 0$, $P(|X - \mu| < \varepsilon) \geq 1 - \frac{\sigma^2}{\varepsilon^2}.$

$$P(0 < X < 150) = P(|X - 75| < 75) \geq 1 - \frac{324}{75^2} = \mathbf{0.9424}.$$

b) A lower bound for $P(30 < X < 120)$.

$$P(30 < X < 120) = P(|X - 75| < 45) \geq 1 - \frac{324}{45^2} = \mathbf{0.84}.$$

c) An upper bound for $P(|X - 75| \geq 30)$.

By Chebyshev's Inequality, for $\varepsilon > 0$, $P(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}.$

$$P(|X - 75| \geq 30) \leq \frac{324}{30^2} = \mathbf{0.36}.$$

19. At Meet the Firms, a recruiter is interviewing candidates for an internship. From past experience, the recruiter believes that only about 20% of the potential candidates have the necessary qualifications. Assume independence.

a) What is the probability that the first suitable job candidate will be found during the fourth interview?

F F F S $\qquad 0.80^3 \cdot 0.20 = \mathbf{0.1024}.$

Geometric distribution, $p = 0.20.$

b) What is the probability that it would take at most 7 interviews to find the first suitable job candidate?

For Geometric distribution,

X = number of independent attempts needed to get the first "success".

$\mathrm{P}(\mathrm{X} > a) = \mathrm{P}(\text{the first } a \text{ attempts are "failures"}) = (1-p)^a, \qquad a = 0, 1, 2, 3, \ldots .$

$\mathrm{P}(\mathrm{X} \le 7) = 1 - \mathrm{P}(\mathrm{X} > 7) = 1 - 0.80^7 = \mathbf{0.7902848}.$

c) What is the probability that the first suitable job candidate will be found during an even-numbered interview?

$\text{P(even)} = \text{P}(2) + \text{P}(4) + \text{P}(6) + \ldots = 0.80^1 \cdot 0.20 + 0.80^3 \cdot 0.20 + 0.80^5 \cdot 0.20 + \ldots$

$$= \sum_{k=0}^{\infty} 0.16 \cdot 0.80^{2k} = 0.16 \cdot \sum_{n=0}^{\infty} 0.64^n = 0.16 \cdot \frac{1}{1-0.64} = \frac{16}{36} = \mathbf{\frac{4}{9}} \approx 0.44444.$$

OR

$\text{P(even)} = 0.80^1 \cdot 0.20 + 0.80^3 \cdot 0.20 + 0.80^5 \cdot 0.20 + \ldots$

$\text{P(odd)} = 0.80^0 \cdot 0.20 + 0.80^2 \cdot 0.20 + 0.80^4 \cdot 0.20 + \ldots$

$\Rightarrow$ P(even) = 0.80 · P(odd). P(odd) = $\frac{5}{4}$ · P(even) = 1.25 · P(even).

$\Rightarrow$ 1 = P(odd) + P(even) = 2.25 · P(even) = $\frac{9}{4}$ · P(even).

P(even) = $\mathbf{\frac{4}{9}}$ ≈ 0.44444.

d) What is the probability that the third suitable job candidate will be the eighth person interviewed?

[7 calls: 2 S's & 5 F's] S

$$\left[\ {}_7C_2 \cdot (0.20)^2 \cdot (0.80)^5\ \right] \cdot 0.20 \approx \mathbf{0.05505}.$$

Negative Binomial distribution, $k = 3$, $p = 0.20$.

e) If the recruiter interviews 10 individuals, what is the probability that there will be exactly 3 suitable job candidates?

$$ {}_{10}C_3 \cdot (0.20)^3 \cdot (0.80)^7 \approx \mathbf{0.20133}.$$

Binomial distribution, $n = 10$, $p = 0.20$.

f) If the recruiter interviews 10 individuals, what is the probability that there will be at most 3 suitable job candidates?

$$ {}_{10}C_0 \cdot (0.20)^0 \cdot (0.80)^{10} + {}_{10}C_1 \cdot (0.20)^1 \cdot (0.80)^9$$

$$+ {}_{10}C_2 \cdot (0.20)^2 \cdot (0.80)^8 + {}_{10}C_3 \cdot (0.20)^3 \cdot (0.80)^7$$

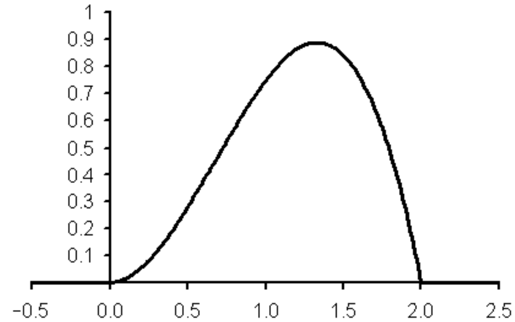
$$\approx 0.10737 + 0.26844 + 0.30199 + 0.20133 = \mathbf{0.87913}.$$

Binomial distribution, $n = 10$, $p = 0.20$.

20. Let X be a continuous random variable with the probability density function

$$f(x) = \frac{3x^2}{2} - \frac{3x^3}{4} \quad \text{for } 0 \le x \le 2,$$

$$f(x) = 0 \quad \text{otherwise.}$$



a) Find $P(X > 1)$.

$$P(X>1) = \int_1^2 \left(\frac{3}{2}x^2 - \frac{3}{4}x^3\right)dx = \left(\frac{1}{2}x^3 - \frac{3}{16}x^4\right)\Bigg|_1^2 = 1 - \frac{5}{16} = \mathbf{\frac{11}{16}} = \mathbf{0.6875}.$$

b) Find $E(X)$.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^2 x \cdot \left(\frac{3}{2}x^2 - \frac{3}{4}x^3\right)dx = \int_0^2 \left(\frac{3}{2}x^3 - \frac{3}{4}x^4\right)dx$$

$$= \left(\frac{3}{8}x^4 - \frac{3}{20}x^5\right)\Bigg|_0^2 = 6 - 4.8 = \mathbf{1.2}.$$

c) Find $SD(X)$.

$$\mathrm{Var}(X) = \left(\int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx\right) - [E(X)]^2 = \int_0^2 x^2 \cdot \left(\frac{3}{2}x^2 - \frac{3}{4}x^3\right)dx - [1.2]^2$$

$$= \int_0^2 \left(\frac{3}{2}x^4 - \frac{3}{4}x^5\right)dx - [1.2]^2 = \left(\frac{3}{10}x^5 - \frac{1}{8}x^6\right)\Bigg|_0^2 - [1.2]^2$$

$$= (9.6 - 8) - 1.44 = 1.6 - 1.44 = 0.16.$$

$$SD(X) = \sqrt{0.16} = \mathbf{0.4}.$$