STAT 400 Spring 2018

Discussion 07 Solutions

1. Suppose that number of accidents at a construction site follows a Poisson process with the average rate of 0.80 accidents per month. Assume all months are independent of each other.

"Hint": If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $F_{T_\alpha}(t) = P(T_\alpha \le t) = P(X_t \ge \alpha)$ and $P(T_\alpha > t) = P(X_t \le \alpha - 1)$, where X_t has a Poisson(λt) distribution.

a) Find the probability that the first accident of a calendar year would occur during March.

T_1 has Exponential distribution with $\lambda = 0.80$ or $\theta = 1/0.80 = 1.25$.

$$P(2 < T_1 < 3) = \int_2^3 0.80\, e^{-0.80t}\, dt = e^{-1.60} - e^{-2.40} \approx \mathbf{0.1112}.$$

OR

$$P(2 < T_1 < 3) = P(T_1 > 2) - P(T_1 > 3) = P(X_2 = 0) - P(X_3 = 0)$$
$$= P(\text{Poisson}(1.60) = 0) - P(\text{Poisson}(2.40) = 0) = 0.202 - 0.091 = \mathbf{0.111}.$$

OR

$$P\begin{pmatrix} \text{no accidents during} & \text{AND} & \text{at least one accident} \\ \text{the first two months} & & \text{during the third month} \end{pmatrix}$$
$$= P(X_2 = 0) \times P(X_1 \ge 1) = 0.202 \times (1 - 0.449) \approx \mathbf{0.111}.$$

OR

Jan	Feb	Mar	
no accident	no accident	accident(s)	
0.449 ×	0.449 ×	0.551	≈ **0.111**.

b) Find the probability that the third accident of a calendar year would occur during April.

T_3 has Gamma distribution with $\alpha = 3$ and $\lambda = 0.80$ or $\theta = 1/0.80 = 1.25$.

$$P(3 < T_3 < 4) = P(T_3 > 3) - P(T_3 > 4) = P(X_3 \le 2) - P(X_4 \le 2)$$

$$= P(\text{Poisson}(2.4) \le 2) - P(\text{Poisson}(3.2) \le 2) = 0.570 - 0.380 = \mathbf{0.190}.$$

OR

$$P(3 < T_3 < 4) = \int_3^4 \frac{0.80^3}{\Gamma(3)} t^{3-1} e^{-0.80t}\, dt = \int_3^4 \frac{0.80^3}{2} t^{3-1} e^{-0.80t}\, dt \approx \mathbf{0.189805}.$$

OR

$$P\left(\begin{array}{c} \text{two accidents during} \\ \text{the first three months} \end{array} \text{ AND } \begin{array}{c} \text{at least one accident} \\ \text{during April} \end{array} \right)$$
$$+ P\left(\begin{array}{c} \text{one accident during} \\ \text{the first three months} \end{array} \text{ AND } \begin{array}{c} \text{at least two accidents} \\ \text{during April} \end{array} \right)$$
$$+ P\left(\begin{array}{c} \text{no accidents during} \\ \text{the first three months} \end{array} \text{ AND } \begin{array}{c} \text{at least three accidents} \\ \text{during April} \end{array} \right) = \ldots$$

c) Find the probability that the third accident of a calendar year would occur during spring (March, April, or May).

T_3 has Gamma distribution with $\alpha = 3$ and $\lambda = 0.80$ or $\theta = 1/0.80 = 1.25$.

$$P(2 < T_3 < 5) = P(T_3 > 2) - P(T_3 > 5) = P(X_2 \le 2) - P(X_5 \le 2)$$

$$= P(\text{Poisson}(1.6) \le 2) - P(\text{Poisson}(4.0) \le 2) = 0.783 - 0.238 = \mathbf{0.545}.$$

OR

$$P(2 < T_3 < 5) = \int_2^5 \frac{0.80^3}{\Gamma(3)} t^{3-1} e^{-0.80t}\, dt = \int_2^5 \frac{0.80^3}{2} t^{3-1} e^{-0.80t}\, dt \approx \mathbf{0.545255}.$$

OR

$$P\left(\begin{matrix}\text{two accidents during}\\ \text{the first two months}\end{matrix} \text{ AND } \begin{matrix}\text{at least one accident}\\ \text{during the next three months}\end{matrix}\right)$$

$$+ P\left(\begin{matrix}\text{one accident during}\\ \text{the first two months}\end{matrix} \text{ AND } \begin{matrix}\text{at least two accidents}\\ \text{during the next three months}\end{matrix}\right)$$

$$+ P\left(\begin{matrix}\text{no accidents during}\\ \text{the first two months}\end{matrix} \text{ AND } \begin{matrix}\text{at least three accidents}\\ \text{during the next three months}\end{matrix}\right) = \ldots$$

2. As Alex is leaving for college, his parents give him a car, but warn him that they would take the car away if Alex gets 6 speeding tickets. Suppose that Alex receives speeding tickets according to Poisson process with the average rate of one ticket per six months.

X_t = number of speeding tickets in t years. Poisson(λt)

T_k = time of the kth speeding ticket. Gamma, $\alpha = k$.

one ticket per six months $\Rightarrow$ $\lambda = 2$.

If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then

$$P(T_\alpha \le t) = P(X_t \ge \alpha) \quad \text{and} \quad P(T_\alpha > t) = P(X_t \le \alpha - 1),$$

where X_t has a Poisson(λt) distribution.

a) Find the probability that it would take Alex longer than two years to get his sixth speeding ticket.

$$P(T_6 > 2) = P(X_2 \le 5) = P(\text{Poisson}(4) \le 5) = \mathbf{0.785}.$$

OR

$$P(T_6 > 2) = \int_2^\infty \frac{2^6}{\Gamma(6)} t^{6-1} e^{-2t}\, dt = \int_2^\infty \frac{2^6}{5!} t^5 e^{-2t}\, dt = \ldots$$

b) Find the probability that it would take Alex less than four years to get his sixth speeding ticket.

$$P(T_6 < 4) = P(X_4 \ge 6) = P(X_4 \ge 6) = 1 - P(X_4 \le 5)$$

$$= 1 - P(\text{Poisson}(8) \le 5) = 1 - 0.191 = \mathbf{0.809}.$$

OR

$$P(T_6 < 4) = \int_0^4 \frac{2^6}{\Gamma(6)} t^{6-1} e^{-2t}\, dt = \int_0^4 \frac{2^6}{5!} t^5 e^{-2t}\, dt = \ldots$$

c) Find the probability that Alex would get his sixth speeding ticket during the fourth year.

$$P(3 < T_6 < 4) = P(T_6 > 3) - P(T_6 > 4) = P(X_3 \le 5) - P(X_4 \le 5)$$
$$= P(\text{Poisson}(6) \le 5) - P(\text{Poisson}(8) \le 5) = 0.446 - 0.191 = \mathbf{0.255}.$$

OR

$$P(3 < T_6 < 4) = \int_3^4 \frac{2^6}{\Gamma(6)} t^{6-1} e^{-2t}\, dt = \int_3^4 \frac{2^6}{5!} t^5 e^{-2t}\, dt = \ldots$$

d) Find the probability that Alex would get his sixth speeding ticket during the third year.

$$P(2 < T_6 < 3) = P(T_6 > 2) - P(T_6 > 3) = P(X_2 \le 5) - P(X_3 \le 5)$$
$$= P(\text{Poisson}(4) \le 5) - P(\text{Poisson}(6) \le 5) = 0.785 - 0.446 = \mathbf{0.339}.$$

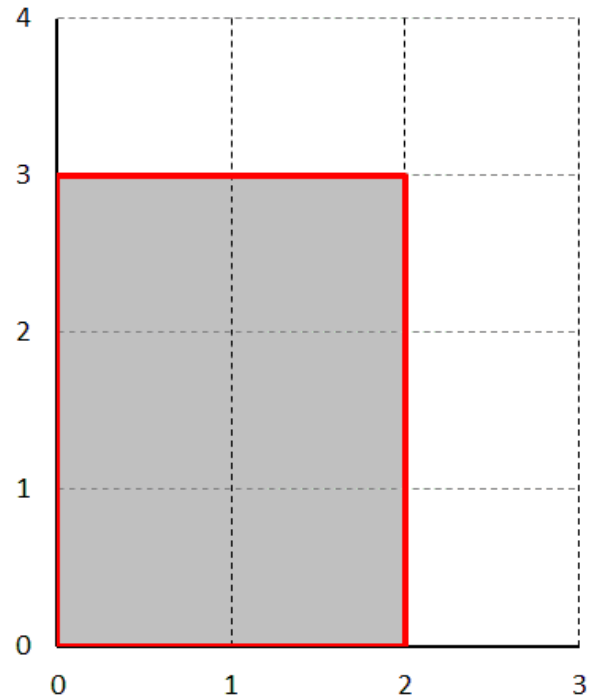
OR

$$P(2 < T_6 < 3) = \int_2^3 \frac{2^6}{\Gamma(6)} t^{6-1} e^{-2t}\, dt = \int_2^3 \frac{2^6}{5!} t^5 e^{-2t}\, dt = \ldots$$

3. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = C(x+2y), \quad 0<x<2, \quad 0<y<3, \quad \text{zero elsewhere.}$$

a) Sketch the support of (X, Y).
That is, sketch
$\{0<x<2, \quad 0<y<3\}$.



b) Find the value of C so that $f_{X,Y}(x,y)$ is a valid joint p.d.f.

$$1 = \int_0^2 \left(\int_0^3 C(x+2y)\,dy \right) dx$$

$$= \int_0^2 \left(C\left(xy+y^2\right) \Big|_0^3 \right) dx$$

$$= \int_0^2 C(3x+9)\,dx$$

$$= C\left(\frac{3}{2}x^2+9x\right)\Big|_0^2 = 24\,C. \qquad \Rightarrow \qquad C = \mathbf{\frac{1}{24}}.$$

c) Find the marginal probability density function of X, $f_X(x)$.

$$f_X(x) = \int_0^3 \frac{1}{24}(x+2y)\,dy = \frac{1}{24}\left(xy+y^2\right)\Big|_0^3 = \frac{1}{24}(3x+9) = \frac{x+3}{8}, \qquad 0<x<2.$$

d) Find the marginal probability density function of Y, $f_Y(y)$.

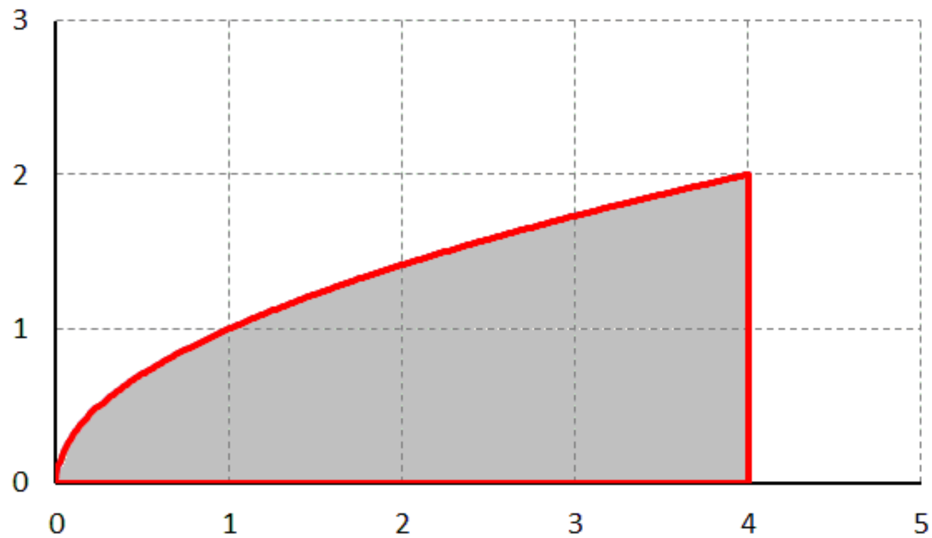
$$f_Y(y) = \int_0^2 \frac{1}{24}(x+2y)\,dx = \frac{1}{24}\left(\frac{x^2}{2}+2xy\right)\Big|_0^2$$

$$= \frac{1}{24}(2+4y) = \frac{1+2y}{12}, \qquad 0<y<3.$$

4. Let X and Y have the joint p.d.f.

$$f_{X,Y}(x,y) = C\,x^2 y, \qquad 0 < x < 4, \quad 0 < y < \sqrt{x}, \qquad \text{zero elsewhere.}$$

a) Sketch the support of (X, Y). That is, sketch $\{\, 0 < x < 4, \ \ 0 < y < \sqrt{x} \,\}$.



b) Find the value of C so that $f_{X,Y}(x,y)$ is a valid joint p.d.f.

$$1 = \int_0^4 \left(\int_0^{\sqrt{x}} C\,x^2\,y\,dy \right) dx = \int_0^4 \frac{C}{2}\,x^2\,y^2 \Big|_0^{\sqrt{x}} dx = \int_0^4 \frac{C}{2}\,x^3\,dx = \frac{C}{8}\,x^4 \Big|_0^4 = 32\,C.$$

$\Rightarrow \qquad C = \mathbf{\dfrac{1}{32}}.$

c) Find the marginal probability density function of X, $f_X(x)$.

$$f_X(x) = \int_0^{\sqrt{x}} \frac{1}{32}\,x^2\,y\,dy = \frac{1}{64}\,x^2\,y^2 \Big|_0^{\sqrt{x}} = \frac{1}{64}\,x^3, \qquad 0 < x < 4.$$

d) Find the marginal probability density function of Y, $f_Y(y)$.

$$f_Y(y) = \int_{y^2}^{4} \frac{1}{32} x^2 y \, dx = \frac{y}{96} \cdot x^3 \Big|_{y^2}^{4} = \frac{y}{96} \cdot \left(64 - y^6\right) = \frac{2}{3} y - \frac{1}{96} y^7, \qquad 0 < y < 2.$$

e) Are X and Y independent?
If X and Y are not independent, find Cov(X, Y).

$f(x, y) \neq f_X(x) \cdot f_Y(y)$. $\Rightarrow$ X and Y are **NOT independent**.

The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

$$E(X) = \int_0^4 x \cdot \frac{1}{64} x^3 \, dx = \int_0^4 \frac{1}{64} x^4 \, dx = \frac{1}{320} x^5 \Big|_0^4 = \frac{16}{5} = 3.2.$$

$$E(Y) = \int_0^2 y \cdot \left(\frac{2}{3} y - \frac{1}{96} y^7 \right) dy = \int_0^2 \left(\frac{2}{3} y^2 - \frac{1}{96} y^8 \right) dy = \left(\frac{2}{9} y^3 - \frac{1}{864} y^9 \right) \Big|_0^2$$

$$= \frac{16}{9} - \frac{16}{27} = \frac{32}{27} \approx 1.1852.$$

$$E(XY) = \int_0^4 \left(\int_0^{\sqrt{x}} x y \cdot \frac{1}{32} x^2 y \, dy \right) dx = \int_0^4 \frac{1}{96} x^3 y^3 \Big|_0^{\sqrt{x}} dx = \int_0^4 \frac{1}{96} x^{9/2} \, dx$$

$$= \frac{1}{528} x^{11/2} \Big|_0^4 = \frac{128}{33} \approx 3.8788.$$

$$\text{Cov}(X, Y) = E(XY) - E(X) \cdot E(Y) = \frac{128}{33} - \frac{16}{5} \cdot \frac{32}{27} = \mathbf{\frac{128}{1485}} \approx 0.0862.$$

5. Let the joint probability density function for (X, Y) be

$$f(x,y) = x + y,$$

$$x > 0, \quad y > 0, \quad x + 2y < 2,$$

zero otherwise.

a) Find the probability $P(Y > X)$.

intersection point:

$y = x$ and $x + 2y = 2$

$x = \frac{2}{3}$ and $y = \frac{2}{3}$

$$P(Y > X) = \int_0^{2/3} \left(\int_x^{1-(x/2)} (x+y)\,dy \right) dx = \int_0^{2/3} \left(\frac{1}{2} + \frac{1}{2}x - \frac{15}{8}x^2 \right) dx = \mathbf{\frac{7}{27}}.$$

OR

$$P(Y > X) = 1 - \int_0^{2/3} \left(\int_y^{2-2y} (x+y)\,dx \right) dy = 1 - \int_0^{2/3} \left(2 - 2y - \frac{3}{2}y^2 \right) dx = \mathbf{\frac{7}{27}}.$$

b) Find the marginal p.d.f. of X, $f_X(x)$.

$$f_X(x) = \int_0^{1-(x/2)} (x+y)\,dy = \frac{1}{2} + \frac{1}{2}x - \frac{3}{8}x^2, \qquad 0 < x < 2.$$

c) Find the marginal p.d.f. of Y, $f_Y(y)$.

$$f_Y(y) = \int_0^{2-2y} (x+y)\,dx = 2-2y, \qquad 0<y<1.$$

d)* Are X and Y independent? If not, find $\text{Cov}(X, Y)$.

The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

OR

$f_{X,Y}(x, y) \neq f_X(x) \times f_Y(y)$. $\Rightarrow$ X and Y are **NOT independent**.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_0^2 x \cdot \left(\frac{1}{2}+\frac{1}{2}x-\frac{3}{8}x^2\right)dx = \int_0^2 \left(\frac{1}{2}x+\frac{1}{2}x^2-\frac{3}{8}x^3\right)dx$$

$$= \left(\frac{1}{4}x^2+\frac{1}{6}x^3-\frac{3}{32}x^4\right)\Bigg|_0^2 = 1+\frac{4}{3}-\frac{3}{2} = \frac{5}{6}.$$

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy = \int_0^1 y \cdot (2-2y)\,dy = \int_0^1 \left(2y-2y^2\right)dy$$

$$= \left(y^2-\frac{2}{3}y^3\right)\Bigg|_0^1 = 1-\frac{2}{3} = \frac{1}{3}.$$

$$E(XY) = \int_0^1 \left(\int_0^{2-2y} xy \cdot (x+y)\,dx\right)dy = \int_0^1 \left(\frac{y}{3}(2-2y)^3+\frac{y^2}{2}(2-2y)^2\right)dx$$

$$= \int_0^1 \left(\frac{8}{3}y-6y^2+4y^3-\frac{2}{3}y^4\right)dy = \frac{4}{3}-2+1-\frac{2}{15} = \frac{1}{5}.$$

$$\text{Cov}(X, Y) = E(XY)-E(X) \times E(Y) = \frac{1}{5}-\frac{5}{6}\times\frac{1}{3} = -\mathbf{\frac{7}{90}} \approx -0.077778.$$

6 – 9. Let the joint probability density function for (X, Y) be

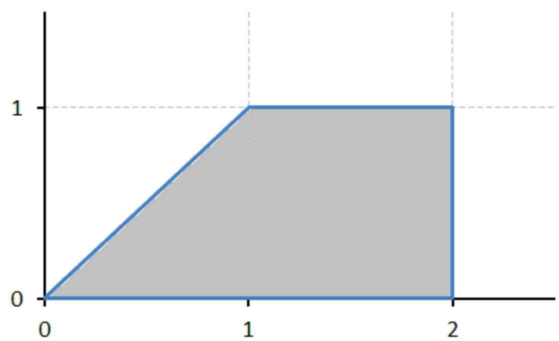
$$f(x,y) = \frac{12}{5} x y^3, \qquad 0 < y < 1, \quad y < x < 2, \qquad \text{zero otherwise.}$$

Do NOT use a computer. You may only use +, –, ×, ÷, and $\sqrt{\ }$ on a calculator. Show all work. Example:

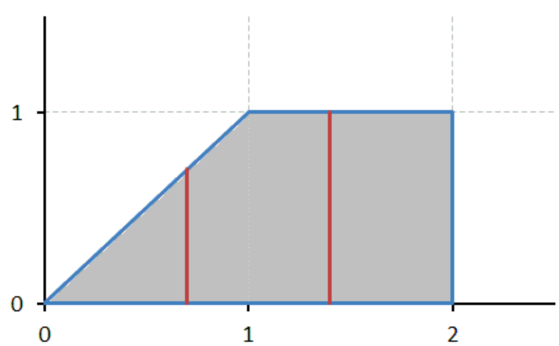
$$\int_0^1 \left(\int_y^2 \frac{12}{5} x y^3 \, dx \right) dy = \int_0^1 \left(\frac{6}{5} x^2 y^3 \right) \Big|_{x=y}^{x=2} dy = \int_0^1 \left(\frac{24}{5} y^3 - \frac{6}{5} y^5 \right) dy$$

$$= \left(\frac{6}{5} y^4 - \frac{1}{5} y^6 \right) \Big|_{y=0}^{y=1} = \frac{6}{5} - \frac{1}{5} = 1. \qquad \Rightarrow \qquad f(x,y) \text{ is a valid joint p.d.f.}$$

6. a) Sketch the support of (X, Y). That is, sketch $\{ 0 < y < 1, \ y < x < 2 \}$.



b) Find the marginal probability density function of X, $f_X(x)$.

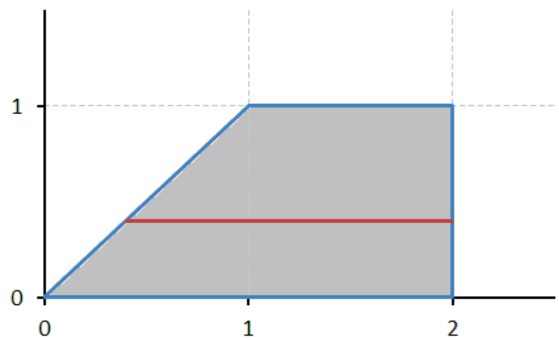


For $0 < x < 1$, $\quad f_X(x) = \int_0^x \frac{12}{5} x y^3 \, dy = \left(\frac{3}{5} x y^4 \right) \Big|_{y=0}^{y=x} = \frac{3}{5} x^5.$

For $1 < x < 2$, $\quad f_X(x) = \int_0^1 \frac{12}{5} x y^3 \, dy = \left(\frac{3}{5} x y^4 \right) \Big|_{y=0}^{y=1} = \frac{3}{5} x.$

Check: $$\int_0^1 \frac{3}{5}x^5\,dx + \int_1^2 \frac{3}{5}x\,dx = \left(\frac{1}{10}x^6\right)\Big|_{x=0}^{x=1} + \left(\frac{3}{10}x^2\right)\Big|_{x=1}^{x=2} = \frac{1}{10}+\frac{9}{10} = 1.$$

c) Find the marginal probability density function of Y, $f_Y(y)$.



For $0 < y < 1$, $$f_Y(y) = \int_y^2 \frac{12}{5}x\,y^3\,dx = \left(\frac{6}{5}x^2y^3\right)\Big|_{x=y}^{x=2} = \frac{24}{5}y^3 - \frac{6}{5}y^5.$$

Check: $$\int_0^1 \left(\frac{24}{5}y^3 - \frac{6}{5}y^5\right)dy = \left(\frac{6}{5}y^4 - \frac{1}{5}y^6\right)\Big|_{y=0}^{y=1} = \frac{6}{5}-\frac{1}{5} = 1.$$

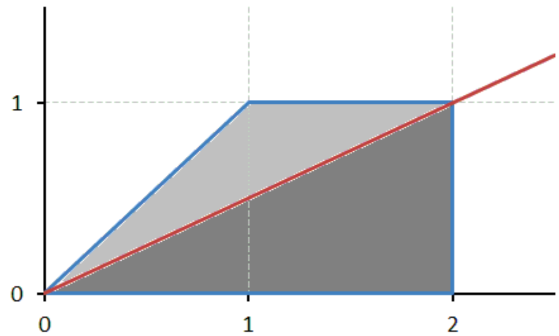
d) Are X and Y independent? Justify your answer.

The support of (X, Y) is NOT a rectangle. X and Y are **NOT independent**.

OR

Since $f(x,y) \neq f_X(x)\cdot f_Y(y)$, X and Y are **NOT independent**.

7. Find the probability $P(X > 2Y)$.



a) Set up the double integral(s) over the region that "we want" with the outside integral w.r.t. x and the inside integral w.r.t. y.

$$\int_0^2 \left(\int_0^{x/2} \frac{12}{5} x\, y^3\, dy \right) dx$$

b) Set up the double integral(s) over the region that "we want" with the outside integral w.r.t. y and the inside integral w.r.t. x.

$$\int_0^1 \left(\int_{2y}^{2} \frac{12}{5} x\, y^3\, dx \right) dy$$

c) Set up the double integral(s) over the region that "we do not want" with the outside integral w.r.t. x and the inside integral w.r.t. y.

$$\int_0^1 \left(\int_{x/2}^{x} \frac{12}{5} x\, y^3\, dy \right) dx + \int_1^2 \left(\int_{x/2}^{1} \frac{12}{5} x\, y^3\, dy \right) dx$$

d) Set up the double integral(s) over the region that "we do not want" with the outside integral w.r.t. y and the inside integral w.r.t. x.

$$\int_0^1 \left(\int_{y}^{2y} \frac{12}{5} x\, y^3\, dx \right) dy$$

e) Use one of (a) – (d) to find the desired probability.

(a) $$\int_0^2 \left(\int_0^{x/2} \frac{12}{5} x\, y^3\, dy \right) dx = \int_0^2 \frac{3}{80} x^5\, dx = \left(\frac{1}{160} x^6 \right) \Bigg|_{x=0}^{x=2} = \mathbf{\frac{2}{5}} = \mathbf{0.40}.$$

(b) $$\int_0^1 \left(\int_{2y}^2 \frac{12}{5} x\, y^3\, dx \right) dy = \int_0^1 \left(\frac{24}{5} y^3 - \frac{24}{5} y^5 \right) dy = \left(\frac{6}{5} y^4 - \frac{4}{5} y^6 \right) \Bigg|_{y=0}^{y=1} = \mathbf{\frac{2}{5}}.$$

(c) $$1 - \int_0^1 \left(\int_{x/2}^{x} \frac{12}{5} x\, y^3\, dy \right) dx - \int_1^2 \left(\int_{x/2}^{1} \frac{12}{5} x\, y^3\, dy \right) dx$$

$$= 1 - \int_0^1 \left(\frac{3}{5} x^5 - \frac{3}{80} x^5 \right) dx - \int_1^2 \left(\frac{3}{5} x - \frac{3}{80} x^5 \right) dx$$

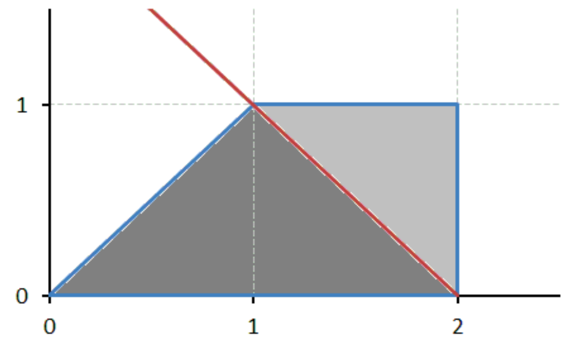
$$= 1 - \left(\frac{1}{10} x^6 - \frac{1}{160} x^6 \right) \Bigg|_{x=0}^{x=1} - \left(\frac{3}{10} x^2 - \frac{1}{160} x^6 \right) \Bigg|_{x=1}^{x=2}$$

$$= 1 - \left(\frac{1}{10} - \frac{1}{160} \right) - \left(\frac{6}{5} - \frac{2}{5} \right) + \left(\frac{3}{10} - \frac{1}{160} \right) = \mathbf{\frac{2}{5}}.$$

(d) $$1 - \int_0^1 \left(\int_y^{2y} \frac{12}{5} x\, y^3\, dx \right) dy = 1 - \int_0^1 \left(\frac{24}{5} y^5 - \frac{6}{5} y^5 \right) dy = 1 - \int_0^1 \frac{18}{5} y^5\, dy$$

$$= 1 - \left(\frac{3}{5} y^6 \right) \Bigg|_{y=0}^{y=1} = 1 - \frac{3}{5} = \mathbf{\frac{2}{5}}.$$

8. Find the probability $P(X+Y<2)$.



a) Set up the double integral(s) over the region that "we want" with the outside integral w.r.t. x and the inside integral w.r.t. y.

$$\int_0^1\left(\int_0^x \frac{12}{5}x\,y^3\,dy\right)dx + \int_1^2\left(\int_0^{2-x} \frac{12}{5}x\,y^3\,dy\right)dx$$

b) Set up the double integral(s) over the region that "we want" with the outside integral w.r.t. y and the inside integral w.r.t. x.

$$\int_0^1\left(\int_y^{2-y} \frac{12}{5}x\,y^3\,dx\right)dy$$

c) Set up the double integral(s) over the region that "we do not want" with the outside integral w.r.t. x and the inside integral w.r.t. y.

$$\int_1^2\left(\int_{2-x}^{1} \frac{12}{5}x\,y^3\,dy\right)dx$$

d) Set up the double integral(s) over the region that "we do not want" with the outside integral w.r.t. y and the inside integral w.r.t. x.

$$\int_0^1\left(\int_{2-y}^{2} \frac{12}{5}x\,y^3\,dx\right)dy$$

e) Use one of (a) – (d) to find the desired probability.

(b) $$\int_0^1 \left(\int_y^{2-y} \frac{12}{5} x y^3 \, dx \right) dy = \int_0^1 \left(\frac{6}{5} x^2 y^3 \right) \Big|_{x=y}^{x=2-y} dy = \int_0^1 \left(\frac{24}{5} y^3 - \frac{24}{5} y^4 \right) dy$$

$$= \left(\frac{6}{5} y^4 - \frac{24}{25} y^5 \right) \Big|_{y=0}^{y=1} = \mathbf{\frac{6}{25}} = \mathbf{0.24}.$$

(d) $$1 - \int_0^1 \left(\int_{2-y}^{2} \frac{12}{5} x y^3 \, dx \right) dy = 1 - \int_0^1 \left(\frac{6}{5} x^2 y^3 \right) \Big|_{x=2-y}^{x=2} dy$$

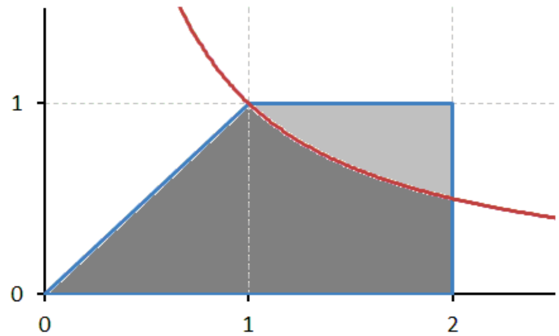
$$= 1 - \int_0^1 \left(\frac{24}{5} y^4 - \frac{6}{5} y^5 \right) dy = 1 - \left(\frac{24}{25} y^5 - \frac{1}{5} y^6 \right) \Big|_{y=0}^{y=1} = \mathbf{\frac{6}{25}}.$$

(a) $$\int_0^1 \left(\int_0^x \frac{12}{5} x y^3 \, dy \right) dx + \int_1^2 \left(\int_0^{2-x} \frac{12}{5} x y^3 \, dy \right) dx$$

$$= \int_0^1 \frac{3}{5} x^5 \, dx + \int_1^2 \frac{3}{5} x (2-x)^4 \, dx = \frac{1}{10} + \int_1^2 \frac{3}{5} x (2-x)^4 \, dx = \ldots$$

(c) $$1 - \int_1^2 \left(\int_{2-x}^{1} \frac{12}{5} x y^3 \, dy \right) dx = 1 - \int_1^2 \left(\frac{3}{5} x - \frac{3}{5} x (2-x)^4 \right) dx = \ldots$$

9. Find the probability $P(XY < 1)$.



a) Set up the double integral(s) over the region that "we want" with the outside integral w.r.t. x and the inside integral w.r.t. y.

$$\int_0^1 \left(\int_0^x \frac{12}{5} x y^3 \, dy \right) dx + \int_1^2 \left(\int_0^{1/x} \frac{12}{5} x y^3 \, dy \right) dx$$

b) Set up the double integral(s) over the region that "we want" with the outside integral w.r.t. y and the inside integral w.r.t. x.

$$\int_0^{1/2} \left(\int_y^2 \frac{12}{5} x y^3 \, dx \right) dy + \int_{1/2}^1 \left(\int_y^{1/y} \frac{12}{5} x y^3 \, dx \right) dy$$

c) Set up the double integral(s) over the region that "we do not want" with the outside integral w.r.t. x and the inside integral w.r.t. y.

$$\int_1^2 \left(\int_{1/x}^1 \frac{12}{5} x y^3 \, dy \right) dx$$

d) Set up the double integral(s) over the region that "we do not want" with the outside integral w.r.t. y and the inside integral w.r.t. x.

$$\int_{1/2}^1 \left(\int_{1/y}^2 \frac{12}{5} x y^3 \, dx \right) dy$$

e) Use one of (a) – (d) to find the desired probability.

(a) $$\int_0^1\left(\int_0^x \frac{12}{5}x\,y^3\,dy\right)dx + \int_1^2\left(\int_0^{1/x} \frac{12}{5}x\,y^3\,dy\right)dx = \int_0^1 \frac{3}{5}x^5\,dx + \int_1^2 \frac{3}{5}\frac{1}{x^3}\,dx$$

$$= \left(\frac{1}{10}x^6\right)\Bigg|_{x=0}^{x=1} + \left(-\frac{3}{10}\frac{1}{x^2}\right)\Bigg|_{x=1}^{x=2} = \frac{1}{10}-\frac{3}{40}+\frac{3}{10} = \mathbf{\frac{13}{40}} = \mathbf{0.325}.$$

(b) $$\int_0^{1/2}\left(\int_y^2 \frac{12}{5}x\,y^3\,dx\right)dy + \int_{1/2}^1\left(\int_y^{1/y} \frac{12}{5}x\,y^3\,dx\right)dy$$

$$= \int_0^{1/2}\left(\frac{24}{5}y^3-\frac{6}{5}y^5\right)dy + \int_{1/2}^1\left(\frac{6}{5}y-\frac{6}{5}y^5\right)dy$$

$$= \left(\frac{6}{5}y^4-\frac{1}{5}y^6\right)\Bigg|_0^{1/2} + \left(\frac{3}{5}y^2-\frac{1}{5}y^6\right)\Bigg|_{1/2}^1$$

$$= \frac{3}{40}-\frac{1}{320}+\frac{3}{5}-\frac{1}{5}-\frac{3}{20}+\frac{1}{320} = \frac{2}{5}-\frac{3}{40} = \mathbf{\frac{13}{40}}.$$

(c) $$1-\int_1^2\left(\int_{1/x}^1 \frac{12}{5}x\,y^3\,dy\right)dx = 1-\int_1^2\left(\frac{3}{5}x-\frac{3}{5}\frac{1}{x^3}\right)dx = 1-\left(\frac{3}{10}x^2+\frac{3}{10}\frac{1}{x^2}\right)\Bigg|_{x=1}^{x=2}$$

$$= 1-\left(\frac{6}{5}+\frac{3}{40}-\frac{3}{10}-\frac{3}{10}\right) = 1-\left(\frac{3}{5}+\frac{3}{40}\right) = 1-\frac{27}{40} = \mathbf{\frac{13}{40}}.$$

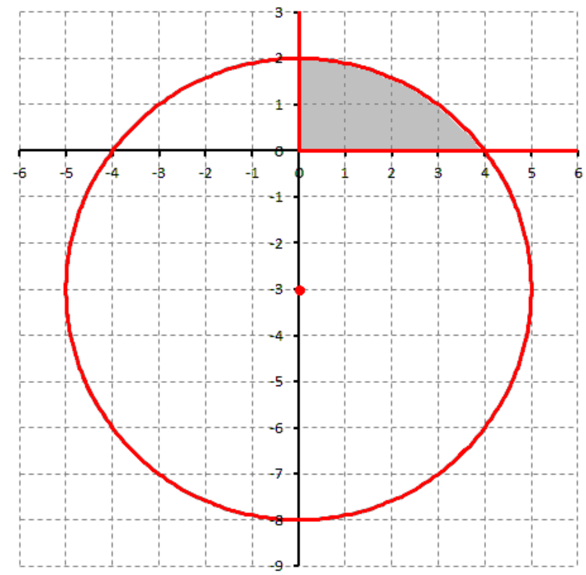
(d) $$1-\int_{1/2}^1\left(\int_{1/y}^2 \frac{12}{5}x\,y^3\,dx\right)dy = 1-\int_{1/2}^1\left(\frac{24}{5}y^3-\frac{6}{5}y\right)dy = 1-\left(\frac{6}{5}y^4-\frac{3}{5}y^2\right)\Bigg|_{1/2}^1$$

$$= 1-\left(\frac{6}{5}-\frac{3}{5}-\frac{3}{40}+\frac{3}{20}\right) = 1-\left(\frac{3}{5}+\frac{3}{40}\right) = 1-\frac{27}{40} = \mathbf{\frac{13}{40}}.$$

10. Let the joint probability density function for (X, Y) be

$$f(x,y) = C\,x\,y,$$
$$x > 0, \qquad y > 0,$$
$$x^2 + (y+3)^2 < 25,$$

zero elsewhere.



a) Find the value of C so that $f(x,y)$ is a valid joint p.d.f.

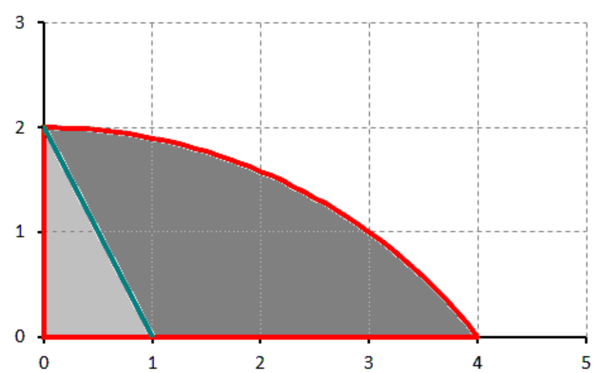
Must have
$$1 = \int_0^2 \left[\int_0^{\sqrt{25-(y+3)^2}} C\,x\,y\,dx \right] dy = \int_0^2 \frac{C}{2}\,y\left[25-(y+3)^2\right]dy$$

$$= \frac{C}{2}\int_0^2 \left[16\,y - 6\,y^2 - y^3\right]dy$$

$$= \frac{C}{2}\left[8\,y^2 - 2\,y^3 - \frac{1}{4}y^4\right]\Bigg|_0^2 = 6\,C.$$

$\Rightarrow$ $\quad \mathrm{C} = \mathbf{\frac{1}{6}}.$

b) Find $\mathrm{P}(2\,\mathrm{X} + \mathrm{Y} > 2)$.



$$1 - \int_0^1 \left(\int_0^{2-2x} \frac{1}{6}\,x\,y\,dy \right) dx$$

$$= 1 - \int_0^1 \frac{1}{12}(2-2x)^2\,x\,dx$$

$$= 1 - \int_0^1 \frac{1}{3}\left(x - 2x^2 + x^3\right)dx = 1 - \frac{1}{3}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = 1 - \frac{1}{36} = \mathbf{\frac{35}{36}}.$$

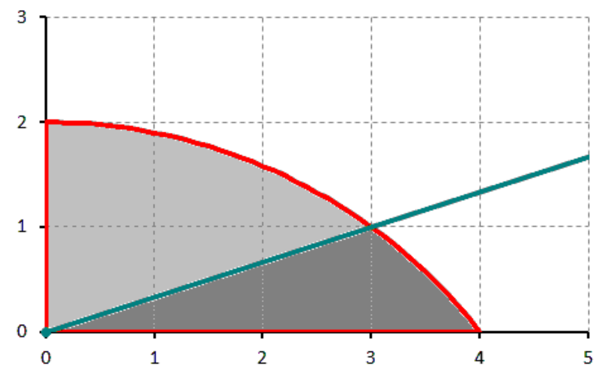
OR

$$1-\int_0^2\left(\int_0^{\frac{2-y}{2}}\frac{1}{6}x\,y\,dx\right)dy = \ldots \qquad \int_0^2\left(\int_{\frac{2-y}{2}}^{\sqrt{25-(y+3)^2}}\frac{1}{6}x\,y\,dx\right)dy = \ldots$$

OR

$$\int_0^1\left(\int_{2-2x}^{-3+\sqrt{25-x^2}}\frac{1}{6}x\,y\,dy\right)dx+\int_1^4\left(\int_0^{-3+\sqrt{25-x^2}}\frac{1}{6}x\,y\,dy\right)dx = \ldots$$

c) Find $P(X-3Y>0)$.



$$P(X-3Y>0) = P(X>3Y)$$

$$= \int_0^1\left[\int_{3y}^{\sqrt{25-(y+3)^2}}\frac{1}{6}x\,y\,dx\right]dy$$

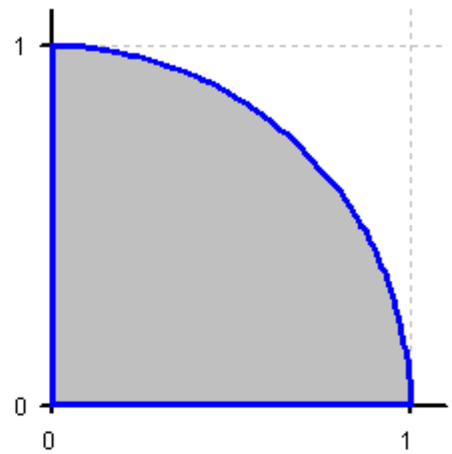
$$= \int_0^1\frac{1}{12}y\left[25-(y+3)^2-9y^2\right]dy = \int_0^1\frac{1}{12}y\left[16-6y-10y^2\right]dy$$

$$= \int_0^1\left[\frac{4}{3}y-\frac{1}{2}y^2-\frac{5}{6}y^3\right]dy = \frac{2}{3}-\frac{1}{6}-\frac{5}{24} = \mathbf{\frac{7}{24}} \approx 0.2916667.$$

11. Suppose that (X, Y) is uniformly distributed over the region defined by $x \geq 0,\ \ y \geq 0,\ \ x^2 + y^2 \leq 1$. That is,

$$f(x, y) = C, \qquad x \geq 0,\ \ y \geq 0,\ \ x^2 + y^2 \leq 1, \qquad\qquad \text{zero elsewhere.}$$

a) What is the joint probability density function of X and Y? That is, find the value of C so that $f(x, y)$ is a valid joint p.d.f.

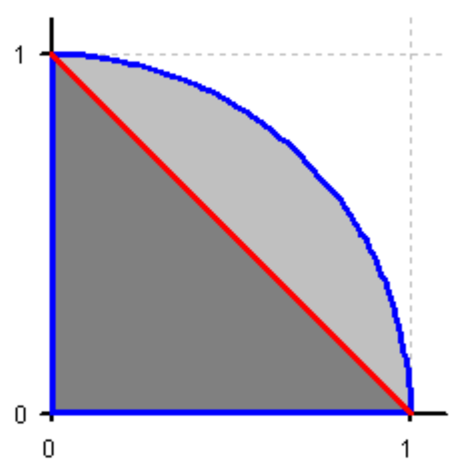


The area of a circle is πr^2.

$\Rightarrow$ The area of the support of (X, Y) is $\dfrac{\pi}{4}$.

$\Rightarrow$ $C = \dfrac{\mathbf{4}}{\boldsymbol{\pi}} \approx 1.27324.$

b) Find $P(X + Y < 1)$.

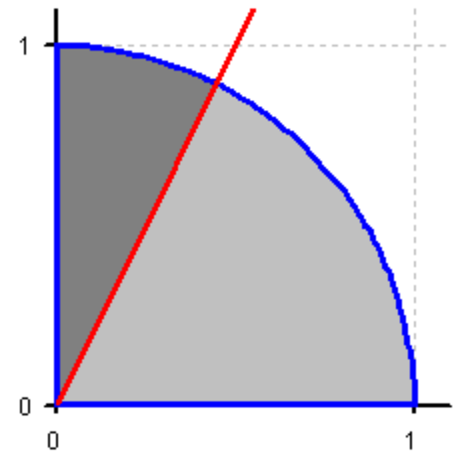


Since uniform,

$$\frac{\textit{want area}}{\textit{total area}} = \frac{\frac{1}{2}}{\frac{\pi}{4}} = \frac{2}{\pi}$$

$\approx 0.63662.$

c) Find $P(Y > 2X)$.



Since uniform,

$$1 - \frac{\arctan(2)}{\frac{\pi}{2}} = 1 - \frac{2 \times \arctan(2)}{\pi}$$

$$\approx 0.295167.$$

OR

$$\frac{\arctan\left(\frac{1}{2}\right)}{\frac{\pi}{2}} = \frac{2 \times \arctan\left(\frac{1}{2}\right)}{\pi}$$

$$\approx 0.295167.$$

d)* Are X and Y independent?

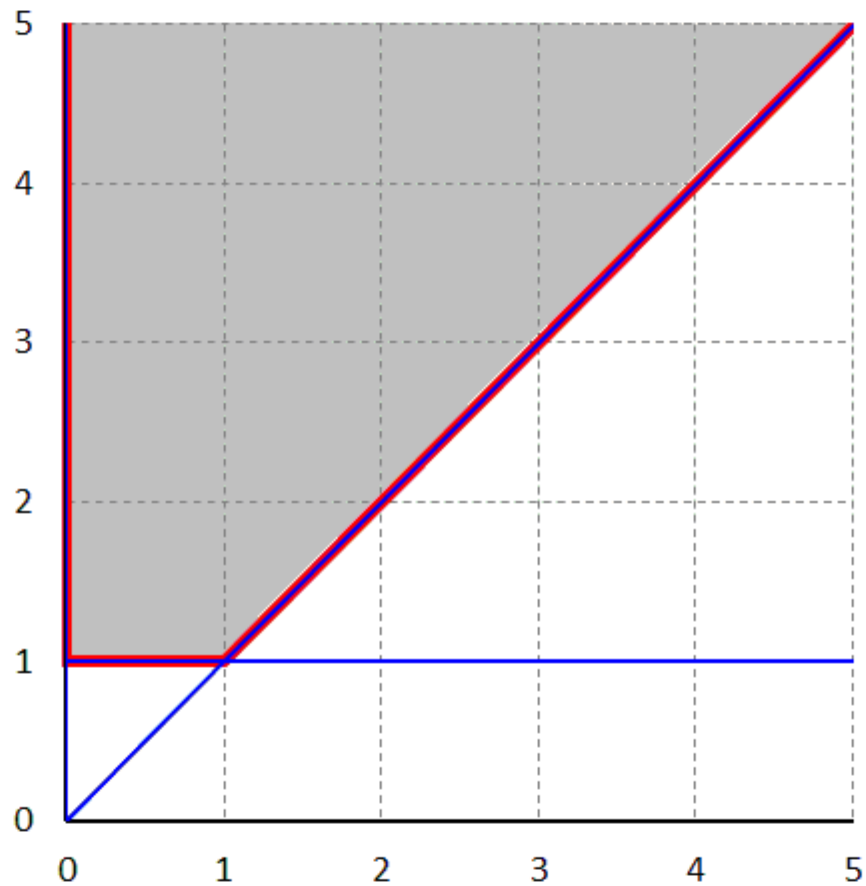
The support of (X, Y) is not a rectangle.

$\Rightarrow$ X and Y are **NOT independent**.

12. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = \frac{C}{(2x+y)^3}, \qquad y>1, \quad 0<x<y, \qquad \text{zero elsewhere.}$$

a) Sketch the support of (X, Y). That is, sketch $\{ y>1, \quad 0<x<y \}$.



b) Find the value of C so that $f_{X,Y}(x,y)$ is a valid joint p.d.f.

$$1 = \int_1^\infty \left(\int_0^y \frac{C}{(2x+y)^3}\,dx \right) dy = \int_1^\infty \left(-\frac{C}{4\,(2x+y)^2} \right) \Bigg|_0^y \,dy$$

$$= \int_1^\infty \left(-\frac{C}{36\,y^2} + \frac{C}{4\,y^2} \right) dy = \frac{2C}{9} \int_1^\infty \frac{1}{y^2}\,dy = \frac{2C}{9}\left(-\frac{1}{y} \right) \Bigg|_1^\infty = \frac{2C}{9}.$$

$$\Rightarrow \quad C = \mathbf{\frac{9}{2}} = \mathbf{4.5}.$$

c) Find the marginal probability density function of X, $f_X(x)$.

For $0 < x < 1$,

$$f_X(x) = \int_1^\infty \frac{9}{2(2x+y)^3}\,dy = -\frac{9}{4\,(2x+y)^2} \Bigg|_1^\infty = \frac{9}{4(2x+1)^2}, \qquad 0 < x < 1.$$

For $1 < x < \infty$,

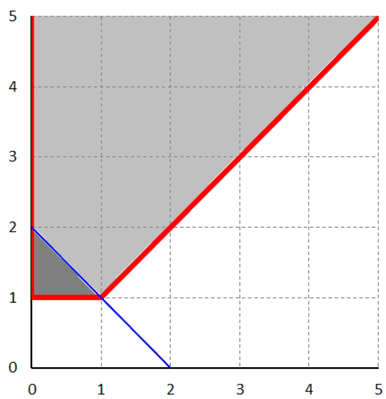
$$f_X(x) = \int_x^\infty \frac{9}{2(2x+y)^3}\,dy = -\frac{9}{4\,(2x+y)^2} \Bigg|_x^\infty = \frac{1}{4x^2}, \qquad 1 < x < \infty.$$

d) Find the marginal probability density function of Y, $f_Y(y)$.

$$f_Y(y) = \int_0^y \frac{9}{2(2x+y)^3}\,dx = -\frac{9}{8\,(2x+y)^2} \Bigg|_0^y$$

$$= -\frac{1}{8y^2} + \frac{9}{8y^2} = \frac{1}{y^2}, \qquad 1 < y < \infty.$$

e) Find $P(X+Y<2)$.



$$P(X+Y<2)$$

$$= \int_0^1 \left(\int_1^{2-x} \frac{9}{2(2x+y)^3}\, dy \right) dx$$

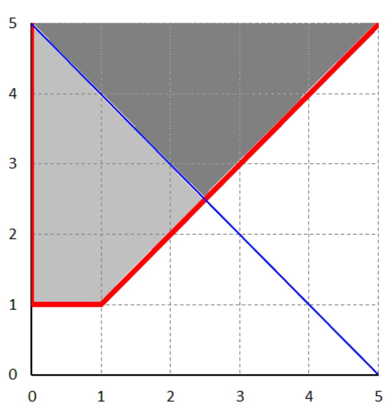
$$= \int_0^1 \left(-\frac{9}{4(2x+y)^2} \right) \Bigg|_1^{2-x} dx$$

$$= \int_0^1 \left(\frac{9}{4(2x+1)^2} - \frac{9}{4(x+2)^2} \right) dx$$

$$= \left(-\frac{9}{8(2x+1)} + \frac{9}{4(x+2)} \right) \Bigg|_0^1$$

$$= -\frac{9}{24} + \frac{9}{12} + \frac{9}{8} - \frac{9}{8} = \mathbf{\frac{3}{8}} = \mathbf{0.375}.$$

f) Find $P(X+Y>5)$.



$$P(X+Y>5)$$

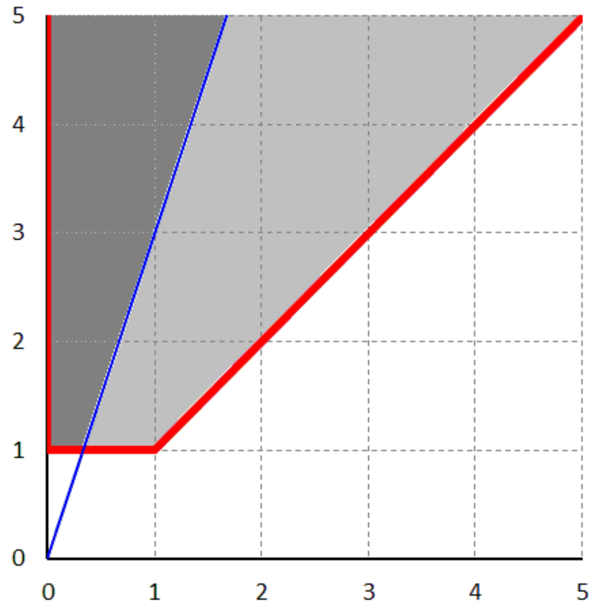
$$= \int_0^{2.5} \left(\int_{5-x}^{\infty} \frac{9}{2(2x+y)^3}\, dy \right) dx + \int_{2.5}^{\infty} \left(\int_x^{\infty} \frac{9}{2(2x+y)^3}\, dy \right) dx$$

$$= \int_0^{2.5} \frac{9}{4(x+5)^2}\, dx + \int_{2.5}^{\infty} \frac{1}{4x^2}\, dx$$

$$= -\frac{9}{4(x+5)} \Bigg|_0^{2.5} - \frac{1}{4x} \Bigg|_{2.5}^{\infty}$$

$$= -\frac{9}{30} + \frac{9}{20} - 0 + \frac{1}{10} = \mathbf{0.25}.$$

g) Find $P(Y > 3X)$.



$P(Y > 3X)$

$$= \int_1^{\infty} \left(\int_0^{y/3} \frac{9}{2(2x+y)^3}\, dx \right) dy$$

$$= \int_1^{\infty} \left(-\frac{9}{8(2x+y)^2} \right) \Bigg|_0^{y/3} dy$$

$$= \int_1^{\infty} \left(\frac{9}{8y^2} - \frac{81}{200y^2} \right) dy$$

$$= \frac{9}{8} - \frac{81}{200} = \mathbf{0.72}.$$

h)* Are X and Y independent?

$f_{X,Y}(x,y) \neq f_X(x) \cdot f_Y(y)$. $\Rightarrow$ X and Y are **NOT independent**.

OR

The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.