STAT 400 Spring 2018

Discussion 09 Solutions

1. Consider a 6-pack of soda. Suppose that the amount of soda in each can follows a normal distribution with mean 12.06 oz and standard deviation 0.15 oz. Assume that all cans are filled independently of each other. Find the probability of the following:

$\mu = 12.06, \qquad \sigma = 0.15.$

a) a can is underfilled, i.e. there is less than 12 oz of soda in a can;

$$P(X < 12) = P\left(Z < \frac{12 - 12.06}{0.15}\right) = P(Z < -0.40) = \Phi(-0.40) = \mathbf{0.3446}.$$

b) all 6 cans are underfilled;

$$(0.3446)^6 = \mathbf{0.001675}. \qquad \text{OR} \qquad {}_6C_6\,(0.3446)^6\,(0.6554)^0 = \mathbf{0.001675}.$$

c) at least one of the 6 cans is underfilled;

$$1 - P(\text{none}) = 1 - (0.6554)^6 = \mathbf{0.920743}.$$

d) exactly 2 of the 6 cans are underfilled;

$${}_6C_2\,(0.3446)^2\,(0.6554)^4 = \mathbf{0.32866}.$$

e) the average amount of soda in these 6 cans is less than 12 oz.

Need $P(\overline{X} < 12) = ?$ $\qquad n = 6 \qquad$ Normal distribution.

$$P(\overline{X} < 12) = P\left(Z < \frac{12 - 12.06}{0.15/\sqrt{6}}\right) = P(Z < -0.98) = \Phi(-0.98) = \mathbf{0.1635}.$$

2. The weights of the eggs at a particular farm are normally distributed with the mean weight of 1.75 oz and standard deviation 0.25 oz.

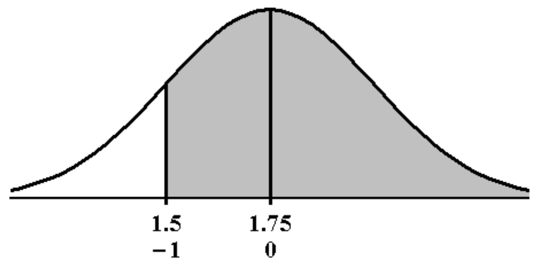
a) Find the probability that a randomly selected egg weighs over 1.5 oz.

$\mu = 1.75, \qquad \sigma = 0.25.$

$$P(X > 1.5) = P\left(Z > \frac{1.5 - 1.75}{0.25}\right)$$

$$= P(Z > -1.00)$$

$$= \mathbf{0.8413}.$$



b) If three dozen eggs (36 eggs) are randomly and independently selected, what is the probability that exactly 30 of them weigh over 1.5 oz?

Let W = the number of egg (out of 36) that weigh over 1.5 oz.
Eggs are selected independently
$\Rightarrow$ Binomial distribution, $n = 36$, $p = 0.8413$ (see part (a)).

Need $P(W = 30) = ?$ $\qquad P(W = k) = {}_nC_k \cdot p^k \cdot (1-p)^{n-k}$

$$P(W = 30) = {}_{36}C_{30} \cdot (0.8413)^{30} \cdot (0.1587)^6$$

$$= 1{,}947{,}792 \cdot (0.8413)^{30} \cdot (0.1587)^6 = \mathbf{0.1744}.$$

c) What is the probability that the total weight of (randomly and independently selected) three dozen eggs (36 eggs) is over 60 oz?

$n = 36.$ Need $P(\text{Total} > 60) = ?$

We sample from a normally distributed population.

$\Rightarrow$ The total weight of (randomly and independently selected) three dozen eggs is normally distributed with mean $36 \times \mu_{1\text{ egg}} = 36 \times 1.75 = 63$ oz and standard deviation $\sqrt{36} \times SD_{1\text{ egg}} = \sqrt{36} \times 0.25 = 1.5$ oz.

$$z = \frac{60 - 63}{1.5} = -2.00. \qquad P(\text{Total} > 60) = P(Z > -2.00) = \mathbf{0.9772}.$$

3. **5.6-8** **5.6-8**

Let X equal the weight in grams of a miniature candy bar. Assume that $\mu = E(X) = 24.43$ and $\sigma^2 = \text{Var}(X) = 2.20$. Let $\overline{X}$ be the sample mean of a random sample of $n = 30$ candy bars. Find:

(a) $E(\overline{X})$. (b) $\text{Var}(\overline{X})$.

(c) $P(24.17 < \overline{X} < 24.82)$, approximately.

(a) $E(\overline{X}) = \mu = 24.43;$

(b) $\text{Var}(\overline{X}) = \dfrac{\sigma^2}{n} = \dfrac{2.20}{30} = 0.0733;$

(c)
$$P(24.17 \le \overline{X} \le 24.82) \approx P\left(\frac{24.17 - 24.43}{\sqrt{0.0733}} \le Z \le \frac{24.82 - 24.43}{\sqrt{0.0733}}\right)$$
$$= P(-0.96 \le Z < 1.44) = 0.7566.$$

4. **5.6-4** **5.6-4**

Approximate $P(39.75 < \overline{X} < 41.25)$, where $\overline{X}$ is the mean of a random sample of size $n = 32$ from a distribution with mean $\mu = 40$ and variance $\sigma^2 = 8$.

$$P(39.75 \le \overline{X} \le 41.25) = P\left(\frac{39.75 - 40}{\sqrt{(8/32)}} \le \frac{\overline{X} - 40}{\sqrt{(8/32)}} \le \frac{41.25 - 40}{\sqrt{(8/32)}}\right)$$
$$\approx P(-0.50 \le Z \le 2.50) = 0.6853.$$

5. The weight of an almond varies with mean 0.047 ounce and standard deviation 0.004 ounce.

a) What is the probability (approximately) that the total weight (of a random sample) of 64 almonds is greater than 3 ounces?

$$\text{E(Total)} = 64 \times 0.047 = 3.008.$$

$$\text{Var(Total)} = 64 \times 0.004^2 = 0.001024. \qquad \text{SD(Total)} = 0.032.$$

$n = 64$ – large. Total is approximately normally distributed.

$$\text{P(Total} > 3) \approx \text{P}\left(Z > \frac{3 - 3.008}{0.032}\right) = \text{P}(Z > -0.25) = \mathbf{0.5987}.$$

b) Determine the sample size (the number of almonds) needed to have the probability of at least 0.90 that the total weight is greater than 16 ounces.

$$\text{P}(Z > -1.28) = 0.8997 \approx 0.90. \qquad \frac{\text{Total} - n \cdot \mu}{\sqrt{n} \cdot \sigma} \approx Z.$$

$$\frac{16 - n \cdot 0.047}{\sqrt{n} \cdot 0.004} \le -1.28. \qquad 0.047 \cdot n - 1.28 \cdot 0.004 \cdot \sqrt{n} - 16 \ge 0.$$

$$\Rightarrow \quad \sqrt{n} = \frac{1.28 \cdot 0.004 \pm \sqrt{1.28^2 \cdot 0.004^2 + 4 \cdot 0.047 \cdot 16}}{2 \cdot 0.047} \approx 18.505 \text{ or } -18.396$$

$$18.505^2 \approx 342.44. \qquad \Rightarrow \qquad n = \textbf{at least 343}. \qquad \text{(round up)}$$

Indeed,

$$n = 342, \qquad \text{P(Total} > 3) \approx \text{P}\left(Z > \frac{16 - 342 \cdot 0.047}{\sqrt{342} \cdot 0.004}\right) \approx \text{P}(Z > -1.00) = 0.8413.$$

$$n = 343, \qquad \text{P(Total} > 3) \approx \text{P}\left(Z > \frac{16 - 343 \cdot 0.047}{\sqrt{343} \cdot 0.004}\right) \approx \text{P}(Z > -1.63) = 0.9484.$$

6. An instructor gives a test to a class containing several hundred students. It is known that the standard deviation of the scores is 14 points. A random sample of 49 scores is obtained.

$\mu = ?$, $\sigma = 14$, $n = 49$.

a) What is the probability that the average score of the students in the sample will differ from the overall average by more than 2 points?

Need $1 - P(\mu - 2 < \overline{X} < \mu + 2) = ?$

$n = 49$ – large. Central Limit Theorem: $\dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} = Z$.

$$1 - P(\mu - 2 < \overline{X} < \mu + 2) \approx 1 - P\left(\frac{(\mu - 2) - \mu}{14 / \sqrt{49}} < Z < \frac{(\mu + 2) - \mu}{14 / \sqrt{49}} \right)$$

$$= 1 - P(-1.00 < Z < 1.00) = \mathbf{0.3174}.$$

b) What is the probability that the average score of the students in the sample will be within 3 points of the overall average?

Need $P(\mu - 3 < \overline{X} < \mu + 3) = ?$

$$P(\mu - 3 < \overline{X} < \mu + 3) \approx P\left(\frac{(\mu - 3) - \mu}{14 / \sqrt{49}} < Z < \frac{(\mu + 3) - \mu}{14 / \sqrt{49}} \right)$$

$$= P(-1.50 < Z < 1.50) = \mathbf{0.8664}.$$

7. Let X_1 and X_2 be independent with normal distributions $N(6, 1)$ and $N(7, 1)$, respectively. Find $P(X_1 > X_2)$.

Hint: Write $P(X_1 > X_2) = P(X_1 - X_2 > 0)$ and determine the distribution of $X_1 - X_2$.

$E(X_1 - X_2) = E(X_1) - E(X_2) = -1.$

$Var(X_1 - X_2) = Var(X_1) + Var(X_2) = 2.$

$X_1 - X_2$ has Normal distribution.

$$P(X_1 > X_2) = P(X_1 - X_2 > 0) = P\left(Z > \frac{0-(-1)}{\sqrt{2}} \right) = P(Z > 0.707) \approx \mathbf{0.24}.$$

8. Compute $P(X_1 + 2X_2 - 2X_3 > 7)$, if X_1, X_2, X_3 are i.i.d. with common distribution $N(1, 4)$.

$E(X_1 + 2X_2 - 2X_3) = E(X_1) + 2E(X_2) - 2E(X_3) = 1.$

$Var(X_1 + 2X_2 - 2X_3) = Var(X_1) + 4\,Var(X_2) + 4\,Var(X_3) = 36.$

$SD(X_1 + 2X_2 - 2X_3) = 6.$

$X_1 + 2X_2 - 2X_3$ has Normal distribution.

$$P(X_1 + 2X_2 - 2X_3 > 7) = P\left(Z > \frac{7-1}{6} \right) = P(Z > 1.00) = \mathbf{0.1587}.$$

9. Let $X_1, X_2, \ldots, X_{70}$ be a random sample of size $n = 70$ from a distribution with p.d.f. $f(x) = \frac{1}{72}(6-x)^2$, $0 < x < 6$, zero elsewhere. Find $P(\overline{X} < 1.6)$ approximately.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_{0}^{6} x \cdot \frac{1}{72}(6-x)^2\, dx = 1.5.$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx = \int_{0}^{6} x^2 \cdot \frac{1}{72}(6-x)^2\, dx = 3.6.$$

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 3.6 - 1.5^2 = 1.35.$$

$n = 70$ – large. $\Rightarrow$ $\dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} \approx Z.$

$$P(\overline{X} < 1.6) \approx P\left(Z < \frac{1.6 - 1.5}{\sqrt{1.35} / \sqrt{70}} \right) = P(Z < 0.72) = \mathbf{0.7642}.$$

10. A machine operation produces widgets whose diameters are normally distributed, with a mean of 4.97 inches and a standard deviation of 0.06 inches. Suppose that specifications specifications require that the widget diameter be 5.00 inches plus or minus 0.12 inches (that is, between 4.88 and 5.12 inches).

a) What proportion of the production will be unacceptable?

$$P(\text{ Acceptable }) = P(4.88 < X < 5.12) = P\left(\frac{4.88 - 4.97}{0.06} < Z < \frac{5.12 - 4.97}{0.06}\right)$$

$$= P(-1.50 < Z < 2.50) = 0.9938 - 0.0668 = 0.9270.$$

P(Unacceptable) = 1 – P(Acceptable) = 1 – 0.9270 = **0.0730**.

b) Suppose 25 widgets are independently and randomly selected from the production process. What is the probability that exactly 2 of the 25 will be unacceptable?

$${}_{25}C_2 \times 0.0730^2 \times 0.9270^{23} = \mathbf{0.279641}.$$ (Binomial Distribution)

c) A quality control inspector selects 25 widgets from the production independently and at random. If the average diameter of the selected widgets is within 0.06 inches of 5.00 inches (that is, between 4.94 and 5.06 inches), the production process is allowed to continue. However, if the average diameter of the selected widgets is not within 0.06 inches of 5.00 inches, the production process is stopped and the machine is checked. What is the probability that the production process will be stopped after examining a random sample of 25 widgets?

$n = 25.$ Need $1 - P(4.94 < \overline{X} < 5.06) = ?$

We sample from a Normal distribution. $\Rightarrow \dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} = Z.$

$$P(4.94 < \overline{X} < 5.06) = P\left(\frac{4.94 - 4.97}{0.06/\sqrt{25}} < Z < \frac{5.06 - 4.97}{0.06/\sqrt{25}}\right) = P(-2.5 < Z < 7.5)$$

$$= 1.000 - 0.0062 = 0.9938.$$

1 – 0.9938 = **0.0062**.

11. 10. (continued)

After a disgruntled employee kicked the machine, the mean shifted to 4.91 inches, and the standard deviation shifted to 0.10 inches, while the distribution of the diameters remained normal.

a) What proportion of the production will be unacceptable after the machine is kicked?

$$P(\text{ Acceptable }) = P(4.88 < X < 5.12) = P\left(\frac{4.88 - 4.91}{0.10} < Z < \frac{5.12 - 4.91}{0.10}\right)$$

$$= P(-0.30 < Z < 2.10) = 0.9821 - 0.3821 = 0.6000.$$

$$P(\text{ Unacceptable }) = 1 - P(\text{ Acceptable }) = 1 - 0.6000 = \mathbf{0.4000}.$$

b) After the machine was kicked, 25 widgets are independently and randomly selected from the production process. What is the probability that at least 7 of the 25 will be unacceptable?

Using Cumulative Binomial Probabilities table or a computer:

$$P(X \ge 7) = 1 - P(X \le 6) = 1 - \text{CDF @ } 6 = 1 - 0.074 = \mathbf{0.926}.$$

c) After the machine was kicked, what is the probability that the production process will be stopped after examining a random sample of 25 widgets?

$n = 25.$ Need $1 - P(4.94 < \overline{X} < 5.06) = ?$

We sample from a Normal distribution. $\Rightarrow$ $\dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} = Z.$

$$P(4.94 < \overline{X} < 5.06) = P\left(\frac{4.94 - 4.91}{0.10/\sqrt{25}} < Z < \frac{5.06 - 4.91}{0.10/\sqrt{25}}\right) = P(1.5 < Z < 7.5)$$

$$= 1.000 - 0.9332 = 0.0668.$$

$1 - 0.0668 = \mathbf{0.9332}.$

12. **6.4-1** **6.4-1**

Let $X_1, X_2, \dots, X_n$ be a random sample from:

$N(\mu, \sigma^2)$ μ unknown, σ known.

Show that $\hat{\mu} = \overline{X}$ is the MLE for μ.

$$L(\mu; x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2\sigma^2}(x_i - \mu)^2 \right\}.$$

$$\ln L(\mu) = -\frac{n}{2}\ln(2\pi) - n\ln(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2 .$$

$$\frac{d}{d\mu}\ln L(\mu) = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu) = 0.$$

$$\sum_{i=1}^{n}(x_i - \mu) = \sum_{i=1}^{n} x_i - n\mu = 0.$$

$$\Rightarrow \qquad \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}.$$

13. 6.4-2 6.4-2

Let $X_1, X_2, \dots, X_n$ be a random sample from:

$N(\mu, \sigma^2)$ μ known, σ unknown.

Show that $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$ is the MLE for σ^2.

$$L\left(\sigma^2; x_1, x_2, \dots, x_n\right) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x_i - \mu)^2\right\}.$$

$$\ln L(\sigma) = -\frac{n}{2}\ln(2\pi) - n\ln(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2.$$

$$\frac{d}{d\sigma}\ln L(\sigma) = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i - \mu)^2 = 0.$$

$$\Rightarrow \qquad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2.$$

OR

$$L\left(\sigma^2; x_1, x_2, \dots, x_n\right) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x_i - \mu)^2\right\}$$

$$= \left(\frac{1}{\sqrt{2\pi\theta}}\right)^n \exp\left\{-\frac{1}{2\theta}\sum_{i=1}^{n}(x_i - \mu)^2\right\}. \qquad \theta = \sigma^2$$

$$\ln L(\theta) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\theta) - \frac{1}{2\theta}\sum_{i=1}^{n}(x_i - \mu)^2.$$

$$\frac{d}{d\theta}\ln L(\theta) = -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum_{i=1}^{n}(x_i - \mu)^2 = 0.$$

$$\Rightarrow \qquad \hat{\theta} = \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2.$$

14. Let $\theta > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with the probability density function

$$f_X(x) = f_X(x;\theta) = \frac{\theta}{2\sqrt{x}}\, e^{-\theta\sqrt{x}}, \qquad x > 0.$$

a) Find the method of moments estimator $\tilde{\theta}$ of θ.

$$E(X) = \int_0^\infty x \cdot f(x)\,dx = \int_0^\infty x \cdot \frac{\theta}{2\sqrt{x}}\, e^{-\theta\sqrt{x}}\,dx$$

$$y = \sqrt{x} \qquad\qquad dy = \frac{dx}{2\sqrt{x}}$$

$$E(X) = \theta \cdot \int_0^\infty y^2 \cdot e^{-\theta y}\,dy$$

Integration by parts:

$$\int_a^b u\,dv = u\,v\Big|_a^b - \int_a^b v\,du$$

Choice of u:

L ogarithmic
A lgebraic
T rigonometric
E xponential

$$u = y^2, \quad dv = \theta \cdot e^{-\theta y}\,dy, \qquad du = 2\,y\,dy, \quad v = -e^{-\theta y}.$$

$$E(X) = -y^2 \cdot e^{-\theta y}\Big|_0^\infty + 2 \cdot \int_0^\infty y \cdot e^{-\theta y}\,dy = 2 \cdot \int_0^\infty y \cdot e^{-\theta y}\,dy$$

$$u = y, \quad dv = e^{-\theta y}\,dy, \qquad du = dy, \quad v = -\frac{1}{\theta} \cdot e^{-\theta y}.$$

$$E(X) = 2 \cdot \left(-y \cdot \frac{1}{\theta} \cdot e^{-\theta y}\Big|_0^\infty + \frac{1}{\theta} \cdot \int_0^\infty e^{-\theta y}\,dy \right) = \frac{2}{\theta} \cdot \int_0^\infty e^{-\theta y}\,dy$$

$$= \frac{2}{\theta} \cdot \left(-\frac{1}{\theta} \cdot e^{-\theta y} \Big|_{0}^{\infty} \right) = \frac{2}{\theta^2}.$$

OR

$$\mathrm{E}(\mathrm{X}) = \int_0^\infty x \cdot f(x)\, dx = \int_0^\infty x \cdot \frac{\theta}{2\sqrt{x}}\, e^{-\theta\sqrt{x}}\, dx$$

$$y = \sqrt{x} \qquad\qquad dy = \frac{dx}{2\sqrt{x}}$$

$$\mathrm{E}(\mathrm{X}) = \int_0^\infty y^2 \cdot \theta e^{-\theta y}\, dy = \mathrm{E}(\mathrm{Y}^2),$$

where Y has Exponential distribution with mean $\frac{1}{\theta}$.

$$\mathrm{E}(\mathrm{X}) = \mathrm{E}(\mathrm{Y}^2) = \mathrm{Var}(\mathrm{Y}) + \left[\mathrm{E}(\mathrm{Y})\right]^2 = \frac{1}{\theta^2} + \left(\frac{1}{\theta}\right)^2 = \frac{2}{\theta^2}.$$

$$\overline{\mathrm{X}} = \frac{1}{n} \cdot \sum_{i=1}^{n} \mathrm{X}_i = \frac{2}{\tilde{\theta}^2} \qquad \Rightarrow \qquad \tilde{\theta} = \sqrt{\frac{2}{\overline{\mathrm{X}}}} = \sqrt{\frac{2 \cdot n}{\sum_{i=1}^{n} \mathrm{X}_i}}$$

b) Suppose $n = 4$, and $x_1 = 0.01$, $x_2 = 0.04$, $x_3 = 0.09$, $x_4 = 0.36$.
Find the method of moments estimate $\tilde{\theta}$ of θ.

$$\sum_{i=1}^{n} \mathrm{X}_i = 0.50. \qquad \bar{x} = 0.125. \qquad \tilde{\theta} = \sqrt{\frac{2}{0.125}} = \mathbf{4}.$$

c) Find the maximum likelihood estimator $\hat{\theta}$ of θ.

$$L(\theta) = \prod_{i=1}^{n} \left(\frac{\theta}{2\sqrt{x_i}} \, e^{-\theta\sqrt{x_i}} \right)$$

$$\ln L(\theta) = n \cdot \ln\theta - \sum_{i=1}^{n} \ln\left(2\sqrt{x_i}\right) - \theta \cdot \sum_{i=1}^{n} \sqrt{x_i}$$

$$(\ln L(\theta))' = \frac{n}{\theta} - \sum_{i=1}^{n} \sqrt{x_i} = 0 \qquad \Rightarrow \qquad \hat{\theta} = \frac{n}{\sum_{i=1}^{n} \sqrt{X_i}}$$

d) Suppose $n = 4$, and $x_1 = 0.01$, $x_2 = 0.04$, $x_3 = 0.09$, $x_4 = 0.36$.

Find the maximum likelihood estimate $\hat{\theta}$ of θ.

$$\sum_{i=1}^{n} \sqrt{x_i} = 1.2. \qquad\qquad \hat{\theta} = \frac{4}{1.2} = \mathbf{\frac{10}{3}} \approx 3.3333.$$

e)* Find a closed-form expression for $E(X^k)$, $k > -\frac{1}{2}$. "Hint": Consider $u = \theta\sqrt{x}$.

$$E(X^k) = \int_0^{\infty} x^k \cdot \frac{\theta}{2\sqrt{x}} \, e^{-\theta\sqrt{x}} \, dx \qquad u = \theta\sqrt{x} \qquad du = \frac{\theta}{2\sqrt{x}} dx$$

$$= \int_0^{\infty} \left(\frac{u}{\theta}\right)^{2k} e^{-u} \, du = \frac{1}{\theta^{2k}} \int_0^{\infty} u^{2k} e^{-u} \, du = \frac{1}{\theta^{2k}} \Gamma(2k+1).$$

For example, $\quad E(X) = E(X^1) = \frac{1}{\theta^2}\Gamma(3) = \frac{2!}{\theta^2} = \frac{2}{\theta^2}.$