STAT 400 Spring 2018

Discussion 10

1. Let $\tau > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\tau) = \frac{\tau^5}{8} x^{14} e^{-\tau x^3}, \qquad x > 0.$$

Obtain the maximum likelihood estimator of τ, $\hat{\tau}$.

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x) = \frac{2(\theta - x)}{\theta^2} \qquad 0 < x < \theta \qquad \theta > 0.$$

a) Obtain the method of moments estimator of θ, $\tilde{\theta}$.

b) Is $\tilde{\theta}$ an unbiased estimator for θ?

c) Find $\operatorname{Var}(\tilde{\theta})$.

3. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the distribution with probability density function

$$f(x;\lambda) = \frac{2\sqrt{\lambda}}{\sqrt{\pi}} e^{-\lambda x^2}, \quad x > 0, \qquad \lambda > 0.$$

a) Obtain the maximum likelihood estimator of λ, $\hat{\lambda}$.

b) Suppose $n = 4$, and $x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 1.1$, $x_4 = 1.7$.
Find the maximum likelihood estimate of λ.

c) Obtain the method of moments estimator of λ, $\tilde{\lambda}$.

d) Suppose $n = 4$, and $x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 1.1$, $x_4 = 1.7$.
Find a method of moments estimate of λ.

e) Find a closed-form expression for $E(X^k)$, $k > -1$.

4. Let $\lambda > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\lambda) = \frac{\lambda^2}{2} e^{-\lambda\sqrt{x}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

a) Find the maximum likelihood estimator of λ, $\hat{\lambda}$.

b) Suppose $n = 4$, and $x_1 = 0.81$, $x_2 = 1.96$, $x_3 = 0.36$, $x_4 = 0.09$.
Find the maximum likelihood estimate of λ, $\hat{\lambda}$.

c) Find a closed-form expression for $E(X^k)$ for $k > -1$.

"Hint" 1: $u = \sqrt{x}$.

"Hint" 2: $\frac{\lambda^\alpha}{\Gamma(\alpha)} u^{\alpha-1} e^{-\lambda u}$ is the p.d.f. of Gamma$(\alpha,\ \theta = \frac{1}{\lambda})$ distribution

d) Find $E(X)$ and $\mathrm{Var}(X)$.

e) Find a method of moments estimator of λ, $\tilde{\lambda}$.

f) Suppose $n = 4$, and $x_1 = 0.81$, $x_2 = 1.96$, $x_3 = 0.36$, $x_4 = 0.09$.
Find a method of moments estimate of λ, $\tilde{\lambda}$.

5. Let $\lambda > 0$ and let $X_1, X_2, \ldots, X_n$ be independent random variables, each with the probability density function

$$f(x;\lambda) = \frac{\lambda}{x^{\lambda+1}}, \qquad x > 1.$$

a) (i) Find the maximum likelihood estimator of λ, $\hat{\lambda}$.

(ii) Suppose $n = 5$, and $x_1 = 1.3$, $x_2 = 1.4$, $x_3 = 2$, $x_4 = 3$, $x_5 = 5$.
Find the maximum likelihood estimate of λ.

b) (i) Find a method of moments estimator of λ, $\tilde{\lambda}$. (Assume $\lambda > 1$.)

(ii) Suppose $n = 5$, and $x_1 = 1.3$, $x_2 = 1.4$, $x_3 = 2$, $x_4 = 3$, $x_5 = 5$.

Find a method of moments estimate of λ.

6. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability mass function

$$P(X_i = 1) = \frac{\theta}{3+\theta}, \quad P(X_i = 2) = \frac{2}{3+\theta}, \quad P(X_i = 3) = \frac{1}{3+\theta}, \qquad \theta > 0.$$

a) Obtain the method of moments estimator $\tilde{\theta}$ of θ.

b) Obtain the maximum likelihood estimator $\hat{\theta}$ of θ.

7. Bert and Ernie find a coin on the sidewalk on Sesame Street. They wish to estimate p, the probability of Heads. Bert got X Heads in N coin tosses (N is fixed, X is random). Ernie got Heads for the first time on the Y^{th} coin toss (Y is random). They decide to combine their information in hope of a better estimate. (Assume independence.)

a) What is the likelihood function $L(p) = L(p; X, N, Y)$?

b) Obtain the maximum likelihood estimator for p.

c) Explain intuitively why your estimator makes good sense.

8. Let $\theta \in$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\theta) = \frac{1}{2} e^{-|x-\theta|}, \qquad x \in .$$

a) Find a method of moments estimator $\tilde{\theta}$ of θ.

b) Find the maximum likelihood estimator $\hat{\theta}$ of θ.

9. A random sample of size $n = 16$ from $N(\mu, \sigma^2 = 64)$ yielded $\overline{x} = 85$.
Construct the following confidence intervals for μ:

a) 95%. b) 90%. c) 80%.

10. What is the minimum sample size required for estimating μ for $N(\mu, \sigma^2 = 64)$ to within ± 3 with confidence level

a) 95%. b) 90%. c) 80%.

11. Suppose the overall (population) standard deviation of the bill amounts at a supermarket is $\sigma = \$13.75$.

a) Find the probability that the sample mean bill amount will be within \$2.00 of the overall mean bill amount for a random sample of 121 customers.

b) What is the minimum sample size required for estimating the overall mean bill amount to within \$2.00 with 95% confidence?

12. 11. (continued)
The supermarket selected a random sample of 121 customers, which showed the sample mean bill amount of \$78.80.

c) Construct a 95% confidence interval for the overall mean bill amount at this supermarket.

d) Suppose the supermarket puts Alex in charge of computing the confidence interval, and he gets the answer $(76.15, 81.45)$. Alex says that he used a different confidence level, but other than that did everything correctly. Find the confidence level used by Alex.