Discussion 11 Solutions

1. A random sample of size $n = 9$ from a normal $N(\mu, \sigma^2)$ distribution is obtained:

4.4 3.7 5.1 4.3 4.7 3.7 3.5 4.6 4.7

a) Compute the sample mean $\bar{x}$ and the sample standard deviation s.

$$\bar{x} = \frac{\sum x}{n} = \frac{4.4 + 3.7 + 5.1 + 4.3 + 4.7 + 3.7 + 3.5 + 4.6 + 4.7}{9} = \frac{38.7}{9} = \mathbf{4.3}.$$

x	x^2
4.4	19.36
3.7	13.69
5.1	26.01
4.3	18.49
4.7	22.09
3.7	13.69
3.5	12.25
4.6	21.16
4.7	22.09
	168.83

OR

x	$x - \bar{x}$	$(x - \bar{x})^2$
4.4	0.1	0.01
3.7	−0.6	0.36
5.1	0.8	0.64
4.3	0	0.00
4.7	0.4	0.16
3.7	−0.6	0.36
3.5	−0.8	0.64
4.6	0.3	0.09
4.7	0.4	0.16
	0	2.42

$$s^2 = \frac{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}{n-1} = \frac{168.83 - \frac{(38.7)^2}{9}}{8}$$

$$= \frac{2.42}{8} = 0.3025.$$

$$s^2 = \frac{\sum\left(x - \bar{x}\right)^2}{n-1} = \frac{2.42}{8} = 0.3025.$$

$$s = \sqrt{s^2} = \sqrt{0.3025} = \mathbf{0.55}.$$

b) Construct a 95% (two-sided) confidence interval for the overall (population) mean.

σ is unknown. $n = 9$ – small. The confidence interval : $\overline{X} \pm t_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$.

$n - 1 = 9 - 1 = \mathbf{8}$ degrees of freedom.

95% confidence level, $\alpha = 0.05$, $\alpha/2 = 0.025$, $t_{\alpha/2}(8) = 2.306$.

$4.3 \pm 2.306 \frac{0.55}{\sqrt{9}}$ $\mathbf{4.30 \pm 0.423}$ $\mathbf{(3.877 ; 4.723)}$

c) Construct a 90% one-sided confidence interval for μ that provides an upper bound for μ.

$t_{0.10}(8) = 1.397$. $\overline{X} + t_{\alpha} \cdot \frac{s}{\sqrt{n}} = 4.3 + 1.397 \cdot \frac{0.55}{\sqrt{9}} = \mathbf{4.556}$.

$\mathbf{(-\infty, 4.556)}$

d) Construct a 95% one-sided confidence interval for μ that provides a lower bound for μ.

$t_{0.05}(8) = 1.860$. $\overline{X} - t_{\alpha} \cdot \frac{s}{\sqrt{n}} = 4.3 - 1.860 \cdot \frac{0.55}{\sqrt{9}} = \mathbf{3.959}$.

$\mathbf{(3.959, \infty)}$

e) Construct a 95% (two-sided) confidence interval for the overall standard deviation.

Confidence Interval for σ^2 :

$$\left(\frac{(n-1)\cdot s^2}{\chi^2_{\alpha/2}}, \frac{(n-1)\cdot s^2}{\chi^2_{1-\alpha/2}} \right).$$

$\alpha = 0.05.$ $\quad \alpha/2 = 0.025.$ $\quad 1 - \alpha/2 = 0.975.$

number of degrees of freedom = $n - 1 = 9 - 1 = 8.$

$\chi^2_{\alpha/2} = 17.54.$ $\quad \chi^2_{1-\alpha/2} = 2.180.$

$$\left(\frac{(9-1)\cdot 0.3025}{17.54}, \frac{(9-1)\cdot 0.3025}{2.180} \right) \qquad (0.13797 \,;\, 1.11009)$$

Confidence Interval for σ : $\left(\sqrt{0.13797}, \sqrt{1.11009} \right) =$ **(0.3714 ; 1.0536)**

f) Construct a 90% one-sided confidence interval for σ that provides an upper bound for σ.

$$\left(0, \sqrt{\frac{(n-1)\cdot s^2}{\chi^2_{1-\alpha}}} \right) \qquad \chi^2_{1-\alpha} = \chi^2_{0.90} = 3.490.$$

$$\left(0, \sqrt{\frac{(9-1)\cdot 0.3025}{3.490}} \right)$$

(0 , 0.8327)

g) Construct a 95% one-sided confidence interval for σ that provides a lower bound for σ.

$$\left(\sqrt{\frac{(n-1)\cdot s^2}{\chi^2_{\alpha}}}, \infty \right) \qquad \chi^2_{\alpha} = \chi^2_{0.05} = 15.51.$$

$$\left(\sqrt{\frac{(9-1)\cdot 0.3025}{15.51}}, \infty \right)$$

(0.395 , ∞)

3. Suppose the time spent on a particular STAT 400 homework follows a normal distribution with an overall standard deviation of 28 minutes and an unknown mean.

a) Suppose a random sample of 49 students is obtained. Find the probability that the average time spent on the homework for students in the sample is within 5 minutes of the overall mean.

$P(\mu - 5 \le \overline{X} \le \mu + 5) = ?$

$n = 49$ – large (plus the distribution we sample from is normal). $\dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} = Z.$

$$P(\mu - 5 \le \overline{X} \le \mu + 5) = P\left(\frac{(\mu - 5) - \mu}{28 / \sqrt{49}} \le Z \le \frac{(\mu + 5) - \mu}{28 / \sqrt{49}} \right)$$

$$= P(-1.25 \le Z \le 1.25) = 0.8944 - 0.1056 = \mathbf{0.7888}.$$

b) A sample of 49 students has a sample mean of 234 minutes spent on the homework. Construct a 90% confidence interval for the overall mean time spent on the homework.

σ is known. The confidence interval : $\overline{X} \pm z_{\alpha/2} \dfrac{\sigma}{\sqrt{n}}.$

$\alpha = 0.10$ $\alpha/2 = 0.05.$ $z_{\alpha/2} = 1.645.$

$234 \pm 1.645 \cdot \dfrac{28}{\sqrt{49}}$ **234 ± 6.58** **(227.42 ; 240.58)**

c) What is the minimum sample size required if we want to estimate the overall mean time spent on the homework to within 5 minutes with 90% confidence?

$$n = \left(\frac{z_{\alpha/2} \cdot \sigma}{\varepsilon} \right)^2 = \left(\frac{1.645 \cdot 28}{5} \right)^2 = 84.86.$$

Round up. $n = \mathbf{85}.$

4. An economist states that 10% of Springfield's labor force is unemployed. A random sample of 400 people in the labor force is obtained, of whom 28 are unemployed.

$n = 400, \quad y = 28, \quad \hat{p} = \frac{y}{n} = \frac{28}{400} = 0.07.$

a) Construct a 95% confidence interval for the unemployment rate in Springfield.

The confidence interval : $\hat{p} \pm z_{\alpha/2} \cdot \sqrt{\frac{\hat{p} \cdot (1-\hat{p})}{n}}.$

95% confidence level $\alpha = 0.05$ $\quad \alpha/2 = 0.025.$ $\quad z_{\alpha/2} = 1.960.$

$0.07 \pm 1.960 \cdot \sqrt{\frac{(0.07)(0.93)}{400}}$ $\quad \mathbf{0.07 \pm 0.025}$ $\quad \mathbf{(0.045\ ,\ 0.095)}$

b) What is the minimum sample size required in order to estimate the unemployment rate in Springfield to within 2% with 95% confidence? (Use the economist's guess.)

Use $p^* = 0.10$ (the economist's guess). $\varepsilon = 0.02.$

$$n = \left(\frac{z_{\alpha/2}}{\varepsilon} \right)^2 \cdot p^* \cdot (1 - p^*) = \left(\frac{1.960}{0.02} \right)^2 \cdot 0.10 \cdot 0.90 = 864.36.$$

Round **up**. $n =$ **865**.

c) What is the minimum sample size required in order to estimate the unemployment rate in Springfield to within 2% with 95% confidence? (Assume no information is available.)

Use $p^* = 0.50$ (since no information is available). $\varepsilon = 0.02.$

$$n = \left(\frac{z_{\alpha/2}}{\varepsilon} \right)^2 \cdot p^* \cdot (1 - p^*) = \left(\frac{1.960}{0.02} \right)^2 \cdot 0.50 \cdot 0.50 = \mathbf{2401}.$$

5. The proportion of defective items is not supposed to be over 15%. A buyer wants to test whether the proportion of defectives exceeds the allowable limit. The buyer takes a random sample of 100 items and finds that 19 are defective.

$$n = 100, \quad y = 19, \quad \hat{p} = \frac{y}{n} = \frac{19}{100} = 0.19.$$

a) Construct a 95% confidence interval for the overall proportion of defective items.

The confidence interval : $\hat{p} \pm z_{\alpha/2} \cdot \sqrt{\frac{\hat{p} \cdot (1-\hat{p})}{n}}$.

95% confidence level $\quad \alpha = 0.05 \quad \alpha/2 = 0.025. \quad z_{\alpha/2} = 1.960.$

$0.19 \pm 1.960 \cdot \sqrt{\frac{(0.19)(0.81)}{100}}$ $\quad$ **0.19 ± 0.077** $\quad$ **(0.113 , 0.267)**

b) What is the minimum sample size required in order to estimate the overall proportion of defective items to within 3% with 95% confidence? (Assume that the overall proportion of defective items is at most 0.20.)

Use $p^* = 0.20$ (the closest to 0.50 possible value). $\quad \varepsilon = 0.03.$

$$n = \left(\frac{z_{\alpha/2}}{\varepsilon} \right)^2 \cdot p^* \cdot (1 - p^*) = \left(\frac{1.960}{0.03} \right)^2 \cdot 0.20 \cdot 0.80 = 682.9511.$$

Round **up**. $\quad n =$ **683**.

6. A coffee machine is regulated so that the amount of coffee dispensed is normally distributed. A random sample of 17 cups is given below:

8.15	7.93	8.04	7.80	8.02	7.92
8.18	7.65	7.73	8.15	7.68	7.85
7.97	7.70	7.75	7.87	8.08	

a) Compute the sample mean and the sample standard deviation.

"Hint": EXCEL =AVERAGE(…) =SUM(…)
=STDEV(…) =VAR(…)

OR R

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> x = c( … )
> mean(x)        > sum(x)
> sd(x)          > var(x)
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$\bar{x}$ = **7.91**. s = **0.175**.

b) Construct a 90% confidence interval for the overall average amount of coffee dispensed by the machine.

$n = 17$. $\alpha = 0.10$.

σ is unknown. The confidence interval : $\overline{\mathrm{X}} \pm t_{\alpha/2} \dfrac{s}{\sqrt{n}}$.

$\alpha/2 = 0.05$. number of degrees of freedom $= n - 1 = 17 - 1 = 16$.

$t_{\alpha/2} = 1.746$. $7.91 \pm 1.746 \cdot \dfrac{0.175}{\sqrt{17}}$ $\mathbf{7.91 \pm 0.074}$ $\mathbf{(7.836, 7.984)}$

7. **7.1-10** **6.2-12**

A leakage test was conducted to determine the effectiveness of a seal designed To keep the inside of a plug airtight. An air needle was inserted into the plug, and the plug and needle were placed under water. The pressure was then increased until leakage was observed. Let X equal the pressure in pounds per square inch. Assume that X follows a normal distribution. The following 10 observations of X were recorded:

3.1 3.3 4.5 2.8 3.5 3.5 3.7 4.2 3.9 3.3

a) Find a point estimate of μ using the observations.

$\bar{x} = \mathbf{3.580}$;

b) Find a point estimate of σ using the observations.

$s^2 = \dfrac{2.356}{9} \approx 0.261778, \qquad s \approx \mathbf{0.51164}$;

c) Find a 95% confidence upper bound for μ.

$t_{0.05}(9) = 1.833,$

$\left(0,\ 3.580 + 1.833 \cdot 0.51164/\sqrt{10}\right) = \mathbf{(0,\ 3.8766)}.$

8. 7. continued

d) Construct a 95% (two-sided) confidence interval for σ.

$$\left(\sqrt{\frac{9 \cdot 0.261778}{19.02}}, \sqrt{\frac{9 \cdot 0.261778}{2.700}}\right) = \mathbf{(0.35195,\ 0.93413)};$$

e) Find a 95% confidence upper bound for σ.

$$\left(0, \sqrt{\frac{9 \cdot 0.261778}{3.325}}\right) = \mathbf{(0,\ 0.84177)};$$

9. Analysis of the venom of seven 8-day-old worker bees yielded the sample mean histamine content (nanograms) of 640, with sample standard deviation of 200. Construct a 90% confidence interval for average histamine content for all worker bees of this age. (Assume that the histamine content is approximately normally distributed.)

σ is unknown

The confidence interval: $\overline{X} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}$.

$n - 1 = 7 - 1 = 6$ degrees of freedom

$\alpha = 0.10$

$t_{0.05} = 1.943$

$640 \pm 1.943 \cdot \frac{200}{\sqrt{7}}$

640 ± 146.877

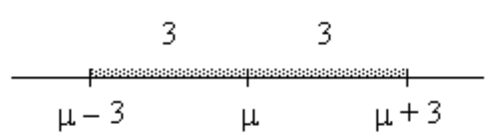
(493.123 ; 786.877)

10. Suppose the IQs of students at Anytown State University are normally distributed with standard deviation 15 and unknown mean.

a) Suppose a random sample of 64 students is obtained. Find the probability that the average IQ of the students in the sample will be within 3 points of the overall mean.

$\sigma = 15.$ $\mu = ?$ $n = 64.$

Need $P(\mu - 3 \le \overline{X} \le \mu + 3) = ?$



$n = 64$ – large (plus the distribution we sample from is normal).

Central Limit Theorem: $\frac{\overline{X} - \mu}{\sigma / \sqrt{n}} = Z.$

$$P(\mu - 3 \le \overline{X} \le \mu + 3) = P\left(\frac{(\mu - 3) - \mu}{15/\sqrt{64}} \le Z \le \frac{(\mu + 3) - \mu}{15/\sqrt{64}} \right)$$

$$= P(-1.60 \le Z \le 1.60) = \Phi(1.60) - \Phi(-1.60)$$

$$= 0.9452 - 0.0548 = \mathbf{0.8904}.$$

b) A sample of 64 students had a sample mean IQ of 115. Construct a 95% confidence interval for the overall mean IQ of students at Anytown State University.

$\overline{X} = 115$ $\sigma = 15$ $n = 64$

σ is known. The confidence interval : $\overline{X} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$.

$\alpha = 0.05$ $\alpha/2 = 0.025.$ $z_{\alpha/2} = 1.96.$

$115 \pm 1.96 \cdot \frac{15}{\sqrt{64}}$ **115 ± 3.675** **(111.325 ; 118.675)**

c) What is the minimum sample size required if we want to estimate the overall mean IQ of students at Anytown State University to within 3 points with 95% confidence?

$\varepsilon = 3.$ $\sigma = 15.$ $\alpha = 0.05.$ $\alpha/2 = 0.025.$ $z_{\alpha/2} = 1.96.$

$n = \left(\frac{z_{\alpha/2} \cdot \sigma}{\varepsilon} \right)^2 = \left(\frac{1.96 \cdot 15}{3} \right)^2 = \mathbf{96.04}.$ Round **up**. $n = \mathbf{97}.$

d) Suppose that only 20% of the students at Anytown State University have the IQ above 130. Find the overall average IQ of the students.

Know $P(X > 130) = 0.20$.

① Find z such that $P(Z > z) = 0.20$.

$\Phi(z) = 0.80.$ $\qquad z = 0.84.$

② $x = \mu + \sigma \cdot z.$ $\qquad 130 = \mu + 15 \cdot (0.84).$ $\qquad \mu =$ **117.4**.

"Hint": From now on, you have μ.

e) Find the probability that the sample average IQ will be 115 or higher for a random sample of 64 students.

Need $P(\overline{X} \geq 115) = ?$

$n = 64$ – large (plus the distribution we sample from is normal).

Central Limit Theorem: $\dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} = Z.$

$$P(\overline{X} \geq 115) = P\left(Z \geq \frac{115 - 117.4}{15/\sqrt{64}} \right) = P(Z \geq -1.28) = 1 - \Phi(-1.28)$$

$$= 1 - 0.1003 = \mathbf{0.8997}.$$

f) Only students in the top 33% are allowed to join the science club. What is the minimum IQ required to be able to join the science club?

Need $x = ?$ such that $P(X > x) = 0.33$.

① Find z such that $P(Z > z) = 0.33$.

$\Phi(z) = 0.67.$ $z = 0.44.$

② $x = \mu + \sigma \cdot z.$ $x = 117.4 + 15 \cdot (0.44) = \mathbf{124}.$

g) What proportion of the students have IQ of 127 or above?

$$P(X \ge 127) = P\left(Z \ge \frac{127 - 117.4}{15}\right) = P(Z \ge 0.64) = 1 - \Phi(0.64)$$

$$= 1 - 0.7389 = \mathbf{0.2611}.$$

h) Find the probability that exactly 13 out of 64 randomly and independently selected students have IQ of 127 or above.

Let Y = number of students (out of the 64 selected) who have IQ of 127 or above.
Then Y has Binomial distribution, $n = 64$, $p = 0.2611$ (see part (g)).

$$P(Y = 13) = \binom{64}{13} 0.2611^{13}\, 0.7389^{51} = \mathbf{0.06837}.$$

11. In a highly publicized study, doctors claimed that aspirin seems to help reduce heart attacks rate. Suppose a group of 400 men from a particular age group took an aspirin tablet three times per week. After three years, 56 of them had had heart attacks. Let p denote the overall proportion of men (in this age group) who take aspirin that have heart attacks in a 3-year period.

a) Construct a 90% confidence interval for p.

$n = 400$, $x = 56$, $\hat{p} = \frac{x}{n} = \frac{56}{400} = 0.14.$

The confidence interval : $\hat{p} \pm z_{\alpha/2} \cdot \sqrt{\frac{\hat{p} \cdot (1-\hat{p})}{n}}$.

$\alpha = 0.10.$ $\alpha/2 = 0.05.$ $z_{\alpha/2} = 1.645.$

$0.14 \pm 1.645 \cdot \sqrt{\frac{(0.14) \cdot (0.86)}{400}}$ **0.14 ± 0.0285** **(0.1115 , 0.1685)**

b) Construct a 95% confidence interval for p.

$\alpha = 0.05.$ $\alpha/2 = 0.025.$ $z_{\alpha/2} = 1.96.$

$0.14 \pm 1.96 \cdot \sqrt{\frac{(0.14) \cdot (0.86)}{400}}$ **0.14 ± 0.034** **(0.106 , 0.174)**

c) Find the 99% confidence upper bound for p.

$\left(0,\ \hat{p} + z_{\alpha} \cdot \sqrt{\frac{\hat{p} \cdot (1-\hat{p})}{n}}\right)$ $\alpha = 0.05.$ $z_{\alpha} = 2.326.$

$\left(0,\ 0.14 + 2.326 \cdot \sqrt{\frac{(0.14)(0.86)}{400}}\right)$ **(0 , 0.18)**

12.* (0) Let X have a $\chi^2(r)$ distribution. If $k > -r/2$, prove (show) that $E(X^k)$ exists and it is given by

$$E(X^k) = \frac{2^k \Gamma\left(\frac{r}{2}+k\right)}{\Gamma\left(\frac{r}{2}\right)}.$$

$$E(X^k) = \int_0^\infty x^k \cdot \frac{1}{\Gamma(r/2)\, 2^{r/2}}\, x^{(r/2)-1}\, e^{-x/2}\, dx$$

$$= \frac{2^k \Gamma\left(\frac{r}{2}+k\right)}{\Gamma\left(\frac{r}{2}\right)} \cdot \int_0^\infty \frac{1}{\Gamma\left(\frac{r}{2}+k\right) 2^{\frac{r}{2}+k}}\, x^{\frac{r}{2}+k-1}\, e^{-x/2}\, dx$$

$$= \frac{2^k \Gamma\left(\frac{r}{2}+k\right)}{\Gamma\left(\frac{r}{2}\right)}, \qquad \text{since } \frac{1}{\Gamma\left(\frac{r}{2}+k\right) 2^{\frac{r}{2}+k}}\, x^{\frac{r}{2}+k-1}\, e^{-x/2}$$

is the p.d.f. of $\chi^2(r+2k)$ distribution.

6.4-14 (a),(b) 6.1-14 (a),(b)

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a $N(\mu, \sigma^2)$ distribution.

(a) Show that an unbiased estimator of σ is $c\,S$, where

$$c = \frac{\sqrt{n-1}\,\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{n}{2}\right)}.$$

Hint: Recall that $X = (n-1)S^2/\sigma^2$ has a $\chi^2(n-1)$ distribution.

"Hint": Select the appropriate value for k in part (0).

From part (0), if $r = n - 1$ and $k = 1/2$, then

$$\mathrm{E}\left(\frac{\sqrt{n-1}\,\mathrm{S}}{\sigma}\right) = \frac{\sqrt{2}\,\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}.$$

Therefore,

$$\mathrm{E}\left(\frac{\sqrt{n-1}\,\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{n}{2}\right)}\cdot \mathrm{S}\right) = \sigma, \quad \text{and} \quad \frac{\sqrt{n-1}\,\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{n}{2}\right)}\cdot \mathrm{S} \text{ is unbiased for } \sigma.$$

(b) Find the value of c when $n = 5$; when $n = 6$.

Recall $\Gamma(x) = (x-1)\,\Gamma(x-1)$.

$\Rightarrow \quad \Gamma(n) = (n-1)!$ if n is an integer.

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$n = 5$ $\quad c = \dfrac{\sqrt{4}\,\Gamma\left(\frac{4}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{5}{2}\right)} = \dfrac{2\cdot 1}{\sqrt{2}\cdot\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}} = \dfrac{8}{3\sqrt{2\pi}} \approx 1.063846.$

$n = 6$ $\quad c = \dfrac{\sqrt{5}\,\Gamma\left(\frac{5}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{6}{2}\right)} = \dfrac{\sqrt{5}\cdot\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}}{\sqrt{2}\cdot 2} = \dfrac{3\sqrt{5\pi}}{8\sqrt{2}} \approx 1.050936.$

$n = 7$ $\quad c \approx 1.042352.$

$n = 8$ $\quad c \approx 1.036237.$

$n = 9$ $\quad c \approx 1.031661.$

$n = 10$ $\quad c \approx 1.028109.$

• • •

$c \to 1$ as $n \to \infty$