STAT 400 AL1
Spring 2018
Exam II
4/19/2018
Time Limit: 75 Minutes
Version: ABC

Last Name: KEY

First Name:

NetID

Discussion CIRCLE BELOW

AD1	**AD2**	**AD3**	**AD4**
Wed 2:00	Wed 4:00	Thu 11:00	Thu 2:00
Neha	Josh	Neha	Josh

This exam contains **14 pages** which include: this cover page, exam material, one page of scratch paper, and a standard normal table. There are a total of **7 problems**. Check to see if any problems are missing. You may detach the tables. Do not turn in your tables if you have detached them. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Write your final answer in the space provided** using **four** significant digits where applicable.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you engage in an act of academic dishonesty, you become liable to severe disciplinary action. Such acts include cheating; falsification or invention of information or citation in an academic endeavor; helping or attempting to help others commit academic infractions; plagiarism; offering bribes, favors, or threats; academic interference; computer related infractions; and failure to comply with research regulations.
- Do not mark on the exam after time has been called.

Problem	Points	Score
1	10	
2	15	
3	11	
4	16	
5	10	
6	15	
7	23	
Total:	100	

1. (10 points) Let $X_1, X_2, \ldots X_n$ be a random sample of size n from a distribution with probability mass function

$$P[X=-1]=\frac{\theta}{2\theta+7}, \quad P[X=0]=\frac{\theta}{2\theta+7}, \quad P[X=1]=\frac{7}{2\theta+7}$$

(a) (8 points) Obtain the method of moments estimator of θ, $\tilde{\theta}$.

$$E[X] = -1\left(\frac{\theta}{2\theta+7}\right) + 0\left(\frac{\theta}{2\theta+7}\right) + 1\left(\frac{7}{2\theta+7}\right)$$

$$= \frac{7-\theta}{2\theta+7}$$

$$\frac{7-\theta}{2\theta+7} = \bar{x} \implies$$

$$\tilde{\theta} = \frac{7-7\bar{x}}{2\bar{x}+1}$$

(a) ____

(b) (2 points) Calculate the method of moments estimate of θ, $\tilde{\theta}$, when

$$x_1 = 0, \ x_2 = 1, \ x_3 = 1, \ x_4 = 1.$$

$$\bar{x} = 3/4 = 0.75$$

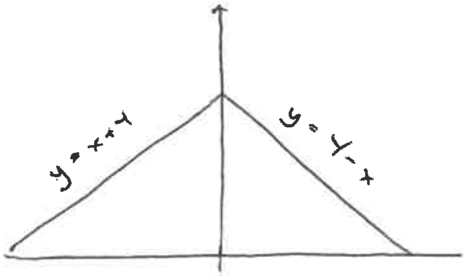
$$\tilde{\theta} = \frac{7-7(0.75)}{2(0.75)+1} =$$

(b) 0.7

2. (15 points) Consider continuous random variables X and Y with joint probability density function

$$f(x,y) = \begin{cases} \frac{3y}{64} & y < x+4,\ y < 4-x,\ y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) (5 points) Find the marginal **distribution** of Y, $f(y)$.

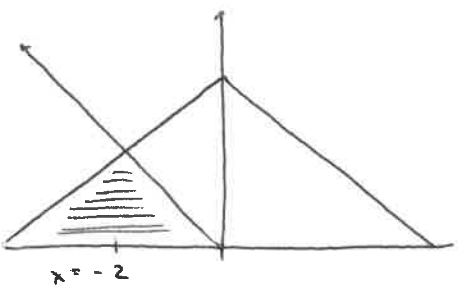


$$f(y) = \int_{y-4}^{4-y} \frac{3y}{64}\,dx = \frac{12y - 3y^2}{32}$$

(a) $f(y) = \frac{12y - 3y^2}{32} \qquad 0 < y < 4$

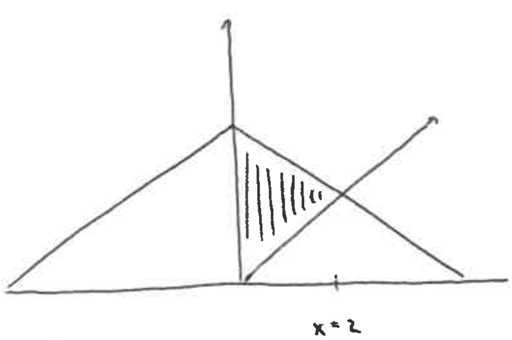
(b) (5 points) Set up the integral(s) needed to calcualte the probability $P[X + Y < 0]$. Do not evaluate the integral.

$-x = x + 4 \Rightarrow x = -2$



(b) $\int_0^2 \int_{y-4}^{-y} \frac{3y}{64}\,dx\,dy$

(c) (5 points) Set up the integral(s) needed to calcualte the probability $P[0 < X < Y]$. Do not evaluate the integral.



(c) $\int_0^2 \int_x^{4-x} \frac{3y}{64}\,dy\,dx$

3. (11 points) Consider discrete random variables X and Y with joint probability mass function

y \ x	3	5	7	9
2	0.1	0.1	0.1	0.0
4	0.0	0.4	0.2	0.1

(a) (3 points) Calculate $P[2Y > X]$.

$P[2Y > X] = 0.1 + 0.0 + 0.4 + 0.2 =$

(a) 0.7

(b) (3 points) Calculate $E[X^2Y]$.

$E[X^2Y] = 3^2 \cdot 2(0.1) + 5^2 \cdot 2(0.1) + 7^2 \cdot 2(0.1) + 0$

$+ 0 + 5^2 \cdot 4(0.4) + 7^2 \cdot 4(0.2) + 9^2 \cdot 4(0.1) =$

(b) 132.8

(c) (3 points) Calculate $P[Y = 4 \mid X = 7]$.

$P[Y=4 \mid X=7] = \dfrac{P[Y=4 \cap X=7]}{P[X=7]} = \dfrac{0.2}{0.1+0.2} =$

(c) $2/3$

(d) (2 points) Are X and Y independent? (Your work should be a justification for your answer.)

$P[X=3 \cap Y=4] = 0 \neq P[X=3]P[Y=4]$

(d) No

1. (23 points) Professor Oak would like to know the average weight, μ, of Snorlaxes. He sends you out into the Pokémon world, and during your journeys, you encounter 10 Snorlaxes. They have an average weight 460 kg with a sample standard deviation of 50 kg. Assume encounters are independent and Snorlax weights follow a Normal distribution.

   (a) (7 points) Construct a 99% confidence interval (CI) for the true mean Snorlax weight, μ.

$n = 10$ $\quad df = 9$ $\quad t_{0.005}(9) = 3.250$

$\bar{x} = 460$

$s = 50$

$\alpha = 0.01$

$\alpha/2 = 0.005$

$\bar{x} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}$

$460 \pm 3.250 \frac{50}{\sqrt{10}}$

460 ± 51.39

(a) $(408.6, 511.4)$

   (b) (7 points) Construct a 90% CI for the true standard deviation of Snorlax weight, σ.

$\alpha = 0.10$ $\quad \chi^2_{0.05}(9) = 16.92$

$\alpha/2 = 0.05$ $\quad \chi^2_{0.95}(9) = 3.325$

$$\left(\sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2}}}, \sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}}\right)$$

$$\left(\sqrt{\frac{9(50)^2}{16.92}}, \sqrt{\frac{9(50)^2}{3.325}}\right)$$

(b) $(36.47, 82.26)$

(c) (3 points) Professor Oak claims that the average weight of Snorlaxes could be at most 430 kg. Calculate the test statistic for the test

$$H_0 : \mu \leq 430 \text{ versus } H_1 : \mu > 430.$$

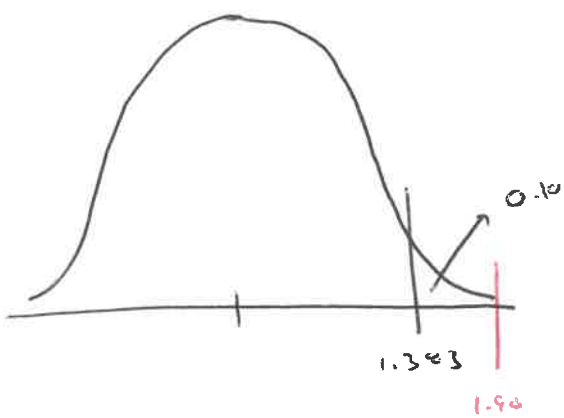
$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{460 - 430}{50/\sqrt{10}} =$$

(c) 1.90

(d) (4 points) Using $\alpha = 0.1$, calculate the critical value for the test

$$H_0 : \mu \leq 430 \text{ versus } H_1 : \mu > 430.$$

$$t_{0.1}(9) = 1.383$$



(d) 1.383

(e) (2 points) State your decision at $\alpha = 0.1$.

(e) REJECT H_0

5. (16 points) Suppose you are interested in the encounter rate, p, of Geodudes in Rock Tunnel. While traveling through Rock Tunnel, you have 120 Pokémon encounters, of which, 42 are Geodudes. Assume encounters are independent.

   (a) (7 points) Construct a 95% confidence interval for the true encounter rate, p.

$$\hat{p} = \frac{42}{120} = 0.35 \qquad \alpha = 0.05 \quad \alpha/2 = 0.025 \qquad z_{0.025} = 1.960$$

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \qquad 0.35 \pm 0.0853$$

$$0.35 \pm 1.960\sqrt{\frac{(0.35)(0.65)}{120}}$$

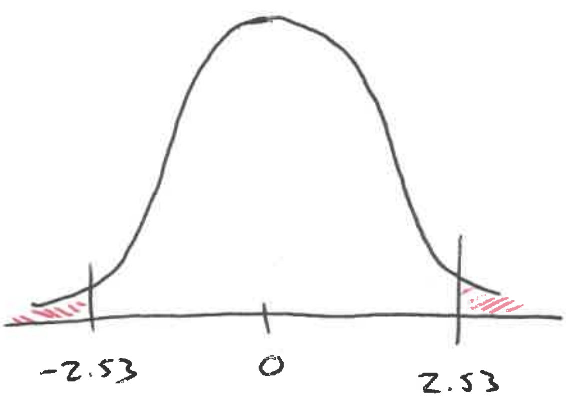
(a) $(0.2647,\ 0.4353)$

   (b) (3 points) Calculate the test statistic for the test $H_0:\ p = 0.25$ versus $H_1:\ p \neq 0.25$.

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.35 - 0.25}{\sqrt{\frac{(0.25)(0.75)}{120}}} =$$

(b) 2.53

   (c) (4 points) Calculate the p-value for the test $H_0:\ p = 0.25$ versus $H_1:\ p \neq 0.25$.



$$\text{P-VAL} = P(Z < -2.53) + P(Z > 2.53)$$
$$= 2 \cdot P(Z < -2.53)$$
$$= 2 \cdot 0.0057 =$$

(c) 0.0114

   (d) (2 points) State your decision at $\alpha = 0.05$.

P-VAL $< \alpha$ $\Longrightarrow$

(d) REJECT H_0

2. (15 points) Suppose that the weight of Squirtle follows a **normal** distribution with a mean of 12 kg and a standard deviation of 2 kg. Also, the weight of Wartortle follows a **normal** distribution with a mean of 22.5 kg and a standard deviation of 3 kg. Lastly, the weight of Blastoise follows an **unknown** distribution with a mean of 87 kg and a standard deviation of 6.25 kg. Last, assume that the weights of these three Pokémon are **independent**.

(a) (5 points) What is the probability that a Wartortle weights more than twice a Squirtle?

$S \sim N(12, 2^2)$

$W \sim N(22.5, 3^2)$

$E[W - 2S] = -1.5$

$V[W - 2S] = 9 + (-2)^2(4) = 25$

$P(W > 2S) = P(W - 2S > 0)$

$= P\left[Z > \frac{0 + 1.5}{5}\right]$

$= P[Z > 0.3] =$

(a) 0.3821

(b) (5 points) The Squirtle Squad is a group of five super cool Squirtle. (You can tell they're cool because they wear sunglasses.) What is the probability that their combined (total) weight exceeds 58 kg?

$Q = S_1 + S_2 + S_3 + S_4 + S_5$

$E[Q] = 5(12) = 60$

$V[Q] = 5(4) = 20$

$P(Q > 58) = P(Z > -0.45) =$

(b) 0.6736

(c) (5 points) Consider a sample of 100 Blastoise. Approximate the probability that their average weight exceeds 87.45 kg?

APPROX VIA CLT

$\bar{X} \sim N(\mu, \sigma^2/n)$

$P[\bar{X} > 87.45] = P\left[Z > \frac{87.45 - 87}{6.25/\sqrt{100}}\right]$

$= P[Z > 0.72] =$

(c) 0.2358

4. (10 points) Let $X_1, X_2, \ldots X_n$ be a random sample of size n from a distribution with probability density function

$$f(x,\alpha) = (5\alpha - 7)e^{7x-5\alpha x}, \quad x > 0,\ \alpha > 7/5$$

(a) (8 points) Obtain the maximum likelihood estimator of α, $\hat{\alpha}$.

$$\mathcal{L}(\alpha) = \prod_{i=1}^{n} (5\alpha - 7) e^{7x_i - 5\alpha x_i} = (5\alpha-7)^n e^{7\sum x_i} e^{-5\alpha \sum x_i}$$

$$\log \mathcal{L}(\alpha) = n \log(5\alpha - 7) + 7\sum x_i - 5\alpha \sum x_i$$

$$\frac{d}{d\alpha} \log \mathcal{L}(\alpha) = \frac{5n}{5\alpha - 7} - 5\sum x_i = 0$$

$$\Rightarrow \frac{5n}{5\alpha-7} = 5\sum x_i$$

$$\Rightarrow \hat{\alpha} = \frac{1}{5}\left(\frac{1}{\bar{x}} + 7\right)$$

(a) $\hat{\alpha} = \frac{1}{5}\left(\frac{1}{\bar{x}} + 7\right)$

(b) (2 points) Calculate the maximum likelihood estimate of α, $\hat{\alpha}$, when

$$x_1 = 1,\ x_2 = 4,\ x_3 = 2,\ x_4 = 1.$$

$$\bar{x} = \frac{8}{4} = 2$$

$$\hat{\alpha} = \frac{1}{5}\left(\frac{1}{2} + 7\right) =$$

(b) 1.5