STAT 400 Homework 09

Spring 2018 | Dalpiaz | UIUC

Due: *Friday, April 6, 2:00 PM*

Exercise 1

Let $X_1, X_2, \ldots X_n$ be a random sample of size n from a distribution with probability density function

$$f(x, \theta) = \frac{1}{\theta} e^{-x/\theta}, \quad x > 0, \ \theta > 0$$

Note that, the moments of this distribution are given by

$$E[X^k] = \int_0^\infty \frac{x^k}{\theta} e^{-x/\theta} = k! \cdot \theta^k.$$

This will be a useful fact for Exercises 2 and 3.

(a) Obtain the maximum likelihood *estimator* of θ, $\hat{\theta}$. (This should be a function of the unobserved x_i and the sample size n.) Calculate the *estimate* when

$$x_1 = 0.50, \ x_2 = 1.50, \ x_3 = 4.00, \ x_4 = 3.00.$$

(This should be a single number, for this dataset.)

Solution:

We first obtain the likelihood by **multiplying** the probability density function for each X_i. We then **simplify** this expression.

$$L(\theta) = \prod_{i=1}^{n} f(x_i; \theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-x_i/\theta} = \theta^{-n} \exp\left(\frac{-\sum_{i=i}^{n} x_i}{\theta} \right)$$

Instead of directly maximizing the likelihood, we instead maximize the **log**-likelihood.

$$\log L(\theta) = -n \log \theta - \frac{\sum_{i=i}^{n} x_i}{\theta}$$

To maximize this function, we take a **derivative** with respect to θ.

$$\frac{d}{d\theta} \log L(\theta) = \frac{-n}{\theta} + \frac{\sum_{i=i}^{n} x_i}{\theta^2}$$

We set this derivative equal to **zero**, then **solve** for θ.

$$\frac{-n}{\theta} + \frac{\sum_{i=i}^{n} x_i}{\theta^2} = 0$$

Solving gives our *estimator*, which we denote with a **hat**.

$$\hat{\theta} = \frac{\sum_{i=i}^{n} x_i}{n} = \bar{x}$$

Using the given data, we obtain an *estimate*.

$$\hat{\theta} = \frac{0.50 + 1.50 + 4.00 + 3}{4} = \boxed{2.25}$$

(b) Calculate the bias of the maximum likelihood *estimator* of θ, $\hat{\theta}$. (This will be a number.)

Solution:

Note that we have an exponential distribution.

$$E[X_i] = \theta$$

$$\text{Var}[X_i] = \theta^2$$

$$\begin{aligned}
\text{Bias}(\hat{\theta}) &= E[\hat{\theta}] - \theta \\
&= E\left[\frac{\sum_{i=i}^{n} X_i}{n}\right] - \theta \\
&= \frac{1}{n}\sum_{i=i}^{n} E[X_i] - \theta \\
&= \frac{1}{n} n\theta - \theta \\
&= \theta - \theta = \boxed{0}
\end{aligned}$$

(c) Find the mean squared error of the maximum likelihood *estimator* of θ, $\hat{\theta}$. (This will be an expression based on the parameter θ and the sample size n. Be aware of your answer to the previous part, as well as the distribution given.)

Solution:

$$\begin{aligned}
\text{MSE}(\hat{\theta}) &= [\text{Bias}(\hat{\theta})]^2 + \text{Var}(\hat{\theta}) \\
&= 0 + \text{Var}\left(\frac{\sum_{i=i}^{n} X_i}{n}\right) \\
&= \frac{1}{n^2}\sum_{i=i}^{n} \text{Var}(X_i) \\
&= \frac{1}{n^2} n\theta^2 = \boxed{\frac{\theta^2}{n}}
\end{aligned}$$

(d) Provide an *estimate* for $P[X > 4]$ when

$$x_1 = 0.50, \; x_2 = 1.50, \; x_3 = 4.00, \; x_4 = 3.00.$$

Solution:

$$P[X > 4] = e^{-4/\theta}$$

$$\hat{P}[X > 4] = e^{-4/\hat{\theta}} = e^{-4/2.25} = \boxed{0.1690}$$

Exercise 2

Let $X_1, X_2, \ldots X_n$ be a random sample of size n from a distribution with probability density function

$$f(x, \alpha) = \alpha^{-2} x e^{-x/\alpha}, \quad x > 0, \ \alpha > 0$$

(a) Obtain the maximum likelihood *estimator* of α, $\hat{\alpha}$. Calculate the *estimate* when

$$x_1 = 0.25, \ x_2 = 0.75, \ x_3 = 1.50, \ x_4 = 2.5, \ x_5 = 2.0.$$

Solution:

We first obtain the likelihood by **multiplying** the probability density function for each X_i. We then **simplify** this expression.

$$L(\alpha) = \prod_{i=1}^{n} f(x_i; \alpha) = \prod_{i=1}^{n} \alpha^{-2} x_i e^{-x_i/\alpha} = \alpha^{-2n} \left(\prod_{i=1}^{n} x_i \right) \exp\left(\frac{-\sum_{i=i}^{n} x_i}{\alpha} \right)$$

Instead of directly maximizing the likelihood, we instead maximize the **log**-likelihood.

$$\log L(\alpha) = -2n \log \alpha + \sum_{i=i}^{n} \log x_i - \frac{\sum_{i=i}^{n} x_i}{\alpha}$$

To maximize this function, we take a **derivative** with respect to α.

$$\frac{d}{d\alpha} \log L(\alpha) = \frac{-2n}{\alpha} + \frac{\sum_{i=i}^{n} x_i}{\alpha^2}$$

We set this derivative equal to **zero**, then **solve** for α.

$$\frac{-2n}{\alpha} + \frac{\sum_{i=i}^{n} x_i}{\alpha^2} = 0$$

Solving gives our *estimator*, which we denote with a **hat**.

$$\hat{\alpha} = \frac{\sum_{i=i}^{n} x_i}{2n} = \frac{\bar{x}}{2}$$

Using the given data, we obtain an *estimate*.

$$\hat{\alpha} = \frac{0.25 + 0.75 + 1.50 + 2.50 + 2.0}{2 \cdot 5} = \boxed{0.70}$$

(b) Obtain the method of moments *estimator* of α, $\tilde{\alpha}$. Calculate the *estimate* when

$$x_1 = 0.25,\ x_2 = 0.75,\ x_3 = 1.50,\ x_4 = 2.5,\ x_5 = 2.0.$$

Solution:

We first obtain the first **population moment**. Notice the integration is done by identifying the form of the integral is that of the second moment of an exponential distribution.

$$E[X] = \int_0^\infty x \cdot \alpha^{-2} x e^{-x/\alpha} dx = \frac{1}{\alpha} \int_0^\infty \frac{x^2}{\alpha} e^{-x/\alpha} dx = \frac{1}{\alpha}(2\alpha^2) = 2\alpha$$

We then set the first population moment, which is a function of α, equal to the first **sample moment**.

$$2\alpha = \frac{\sum_{i=i}^n x_i}{n}$$

Solving for α, we obtain the method of moments *estimator*.

$$\tilde{\alpha} = \frac{\sum_{i=i}^n x_i}{2n} = \frac{\bar{x}}{2}$$

Using the given data, we obtain an *estimate*.

$$\tilde{\alpha} = \frac{0.25 + 0.75 + 1.50 + 2.50 + 2.0}{2 \cdot 5} = \boxed{0.70}$$

Note that, in this case, the MLE and MoM estimators are the same.

Exercise 3

Let $X_1, X_2, \ldots X_n$ be a random sample of size n from a distribution with probability density function

$$f(x, \beta) = \frac{1}{2\beta^3} x^2 e^{-x/\beta}, \quad x > 0,\ \beta > 0$$

(a) Obtain the maximum likelihood *estimator* of β, $\hat{\beta}$. Calculate the *estimate* when

$$x_1 = 2.00,\ x_2 = 4.00,\ x_3 = 7.50,\ x_4 = 3.00.$$

Solution:

We first obtain the likelihood by **multiplying** the probability density function for each X_i. We then **simplify** this expression.

$$L(\beta) = \prod_{i=1}^n f(x_i; \beta) = \prod_{i=1}^n \frac{1}{2\beta^3} x^2 e^{-x/\beta} = 2^{-n} \beta^{-3n} \left(\prod_{i=1}^n x_i \right) \exp\left(\frac{-\sum_{i=i}^n x_i}{\beta} \right)$$

Instead of directly maximizing the likelihood, we instead maximize the **log**-likelihood.

$$\log L(\beta) = -n \log 2 - 3n \log \beta + \sum_{i=i}^n \log x_i - \frac{\sum_{i=i}^n x_i}{\beta}$$

To maximize this function, we take a **derivative** with respect to β.

$$\frac{d}{d\beta} \log L(\beta) = \frac{-3n}{\beta} + \frac{\sum_{i=i}^{n} x_i}{\beta^2}$$

We set this derivative equal to **zero**, then **solve** for β.

$$\frac{-3n}{\beta} + \frac{\sum_{i=i}^{n} x_i}{\beta^2} = 0$$

Solving gives our *estimator*, which we denote with a **hat**.

$$\hat{\beta} = \frac{\sum_{i=i}^{n} x_i}{3n} = \frac{\bar{x}}{3}$$

Using the given data, we obtain an *estimate*.

$$\hat{\beta} = \frac{2.00 + 4.00 + 7.50 + 3.00}{3 \cdot 4} = \boxed{1.375}$$

(b) Obtain the method of moments *estimator* of β, $\tilde{\beta}$. Calculate the *estimate* when

$$x_1 = 2.00,\ x_2 = 4.00,\ x_3 = 7.50,\ x_4 = 3.00.$$

Solution:

We first obtain the first **population moment**. Notice the integration is done by identifying the form of the integral is that of the third moment of an exponential distribution.

$$E[X] = \int_0^\infty x \cdot \frac{1}{2\beta^3} x^2 e^{-x/\beta} dx = \frac{1}{2\beta^2} \int_0^\infty \frac{x^3}{\beta} e^{-x/\beta} dx = \frac{1}{2\beta^2}(6\beta^3) = 3\beta$$

We then set the first population moment, which is a function of β, equal to the first **sample moment**.

$$3\beta = \frac{\sum_{i=i}^{n} x_i}{n}$$

Solving for β, we obtain the method of moments *estimator*.

$$\tilde{\beta} = \frac{\sum_{i=i}^{n} x_i}{3n} = \frac{\bar{x}}{3}$$

Using the given data, we obtain an *estimate*.

$$\tilde{\beta} = \frac{2.00 + 4.00 + 7.50 + 3.00}{3 \cdot 4} = \boxed{1.375}$$

Note again, the MLE and MoM estimators are the same.

Exercise 4

Let $X_1, X_2, \ldots X_n$ be a random sample of size n from a distribution with probability density function

$$f(x, \lambda) = \lambda x^{\lambda - 1}, \quad 0 < x < 1, \lambda > 0$$

(a) Obtain the maximum likelihood *estimator* of λ, $\hat{\lambda}$. Calculate the *estimate* when

$$x_1 = 0.10,\ x_2 = 0.20,\ x_3 = 0.30,\ x_4 = 0.40.$$

Solution:

We first obtain the likelihood by **multiplying** the probability density function for each X_i. We then **simplify** this expression.

$$L(\lambda) = \prod_{i=1}^{n} f(x_i; \lambda) = \prod_{i=1}^{n} \lambda x_i^{\lambda - 1} = \lambda^n \left(\prod_{i=1}^{n} x_i \right)^{\lambda - 1}$$

Instead of directly maximizing the likelihood, we instead maximize the **log**-likelihood.

$$\log L(\lambda) = n \log \lambda + (\lambda - 1) \sum_{i=i}^{n} \log x_i$$

To maximize this function, we take a **derivative** with respect to λ.

$$\frac{d}{d\lambda} \log L(\lambda) = \frac{n}{\lambda} + \sum_{i=i}^{n} \log x_i$$

We set this derivative equal to **zero**, then **solve** for β.

$$\frac{n}{\lambda} + \sum_{i=i}^{n} \log x_i = 0$$

Solving gives our *estimator*, which we denote with a **hat**.

$$\hat{\lambda} = -\frac{n}{\sum_{i=i}^{n} \log x_i}$$

Using the given data, we obtain an *estimate*.

$$\hat{\lambda} = -\frac{n}{\sum_{i=i}^{n} \log x_i} = -\frac{4}{\log(0.1 \cdot 0.2 \cdot 0.3 \cdot 0.4)} = \boxed{0.6631}$$

Note that this is actually a reparameterization of an example seen in class where $\lambda = \frac{1}{\theta}$. Had you realized this, you could have simply found the answer via invariance.

(b) Obtain the method of moments *estimator* of λ, $\tilde{\lambda}$. Calculate the *estimate* when

$$x_1 = 0.10,\ x_2 = 0.20,\ x_3 = 0.30,\ x_4 = 0.40.$$

Solution:

We first obtain the first **population moment**.

$$E[X] = \int_0^1 x \cdot \lambda x^{\lambda-1} dx = \frac{\lambda}{\lambda+1}$$

We then set the first population moment, which is a function of β, equal to the first **sample moment**.

$$\frac{\lambda}{\lambda+1} = \frac{\sum_{i=i}^{n} x_i}{n} = \bar{x}$$

Solving for λ, we obtain the method of moments *estimator*.

$$\tilde{\lambda} = \frac{\bar{x}}{1-\bar{x}}$$

Using the given data, we obtain an *estimate*.

$$\bar{x} = \frac{0.1 + 0.2 + 0.3 + 0.4}{4} = 0.25$$

$$\tilde{\lambda} = \frac{0.25}{1-0.25} = \boxed{\frac{1}{3}}$$

Note that the MLE and MoM *estimators* are different.