STAT 400
UIUC

Practice Problems #7

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The following are a number of practice problems that may be *helpful* for completing the homework, and will likely be **very useful** for studying for exams.

1. Consider two continuous random variables X and Y with joint p.d.f.

$$f(x,y) = \begin{cases} \frac{2}{81} x^2 y & 0 < x < K,\ 0 < y < K \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of K so that $f(x,y)$ is a valid joint p.d.f.

b) Find $P(X > 3Y)$.

c) Find $P(X + Y > 3)$.

d) Are X and Y independent? If not, find $\text{Cov}(X, Y)$.

2. Let X denote the number of times a photocopy machine will malfunction: 0, 1, 2, or 3 times, on any given month. Let Y denote the number of times a technician is called on an emergency call. The joint p.m.f. $p(x,y)$ is presented in the table below:

	x			
y	0	1	2	3
0	0.15	0.30	0.05	0
1	0.05	0.15	0.05	0.05
2	0	0.05	0.10	0.05

a) Find the probability $P(Y > X)$.

b) Find $p_X(x)$, the marginal p.m.f. of X.

c) Find $p_Y(y)$, the marginal p.m.f. of Y.

d) Are X and Y independent? If not, find $\text{Cov}(X, Y)$.

3. Let the joint probability density function for (X, Y) be

$$f(x,y) = \frac{x+y}{2}, \quad x>0, \quad y>0, \quad 3x+y<3, \qquad \text{zero otherwise.}$$

a) Find the probability $P(X < Y)$.

b) Find the marginal probability density function of X, $f_X(x)$.

c) Find the marginal probability density function of Y, $f_Y(y)$.

d) Are X and Y independent? If not, find $\text{Cov}(X, Y)$.

4. Let the joint probability density function for (X, Y) be

$$f(x,y) = \frac{x+y}{3}, \qquad 0<x<2, \quad 0<y<1, \qquad \text{zero otherwise.}$$

a) Find the probability $P(X > Y)$.

b) Find the marginal probability density function of X, $f_X(x)$.

c) Find the marginal probability density function of Y, $f_Y(y)$.

d) Are X and Y independent? If not, find $\text{Cov}(X, Y)$.

5. Two components of a laptop computer have the following joint probability density function for their useful lifetimes X and Y (in years):

$$f(x,y) = \begin{cases} x\,e^{-x(1+y)} & x \ge 0,\ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the marginal probability density function of X, $f_X(x)$.

b) Find the marginal probability density function of Y, $f_Y(y)$.

c) What is the probability that the lifetime of at least one component exceeds 1 year (when the manufacturer’s warranty expires)?

6. Let the joint probability density function for (X, Y) be

$$f(x,y) = \frac{1}{2} e^{-y}, \quad 0 < y < \infty, \quad -y < x < y, \qquad \text{zero otherwise.}$$

a) Find the marginal probability density function of X, $f_X(x)$.

b) Find the marginal probability density function of Y, $f_Y(y)$.

c) Are X and Y independent? If not, find $\text{Cov}(X, Y)$.

7. Suppose Jane has a fair 4-sided die, and Dick has a fair 6-sided die. Each day, they roll their dice at the same time (independently) until someone rolls a "1". (Then the person who did not roll a "1" does the dishes.) Find the probability that …

a) they roll the first "1" at the same time (after equal number of attempts);

b) Dick rolls the first "1" before Jane does.

8. Dick and Jane have agreed to meet for lunch between noon (0:00 p.m.) and 1:00 p.m. Denote Jane's arrival time by X, Dick's by Y, and suppose X and Y are independent with probability density functions

$$f_X(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the probability that Jane arrives before Dick. That is, find $P(X < Y)$.

b) Find the expected amount of time Jane would have to wait for Dick to arrive.

Hint 1: If Dick arrives first (that is, if $X > Y$), then Jane's waiting time is zero. If Jane arrives first (that is, if $X < Y$), then her waiting time is $Y - X$.

Hint 2: $E(g(X, Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) \cdot f(x,y)\, dx\, dy$

9. Suppose that (X, Y) is uniformly distributed over the region defined by $-1 \le x \le 1$ and $0 \le y \le 1 - x^2$. That is,

$$f(x, y) = C, \qquad -1 \le x \le 1,\ 0 \le y \le 1 - x^2, \qquad \text{zero elsewhere.}$$

a) What is the joint probability density function of X and Y? That is, find C.

b) Find the marginal probability density function of X, $f_X(x)$.

c) Find the marginal probability density function of Y, $f_Y(y)$.

10. Let $T_1, T_2, \dots, T_k$ be independent Exponential random variables.

Suppose $E(T_i) = \frac{1}{\lambda_i}, \quad i = 1, 2, \dots, k.$

That is, $f_{T_i}(t) = \lambda_i e^{-\lambda_i t}, \quad t > 0, \quad i = 1, 2, \dots, k.$

Denote $T_{\min} = \min(T_1, T_2, \dots, T_k)$.

a) Show that $T_{\min}$ also has an Exponential distribution. What is the mean of $T_{\min}$?

Hint: Consider $P(T_{\min} > t) = P(T_1 > t \text{ AND } T_2 > t \text{ AND } \dots \text{ AND } T_k > t)$.

b) Find $P(T_1 = T_{\min}) = P(T_1 \text{ is the smallest of } T_1, T_2, \dots, T_k)$

$$= P(T_1 < T_2 \text{ AND } \dots \text{ AND } T_1 < T_k).$$

"Hint": A good place to start is to consider T_1, T_2 and show that $P(T_1 < T_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$.