STAT 400
UIUC

Practice Problems #7

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SOLUTIONS

The following are a number of practice problems that may be *helpful* for completing the homework, and will likely be **very useful** for studying for exams.

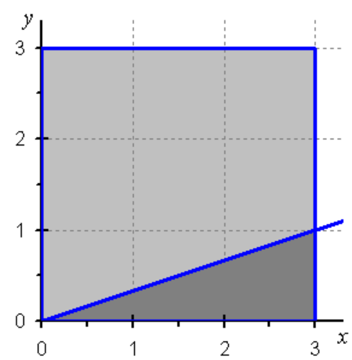
1. Consider two continuous random variables X and Y with joint p.d.f.

$$f(x,y) = \begin{cases} \frac{2}{81} x^2 y & 0 < x < K,\ 0 < y < K \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of K so that $f(x,y)$ is a valid joint p.d.f.

$$1 = \int_0^K \int_0^K \frac{2}{81} x^2 y \, dx \, dy = \frac{K^5}{243}. \qquad \Rightarrow \qquad K = \mathbf{3}.$$

b) Find $P(X > 3Y)$.



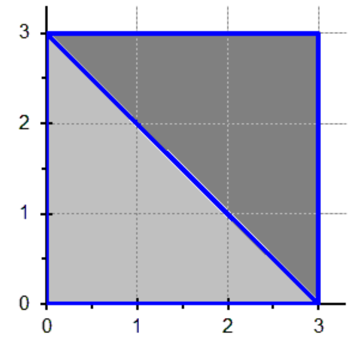
$$P(X > 3Y) = \int_0^3 \left(\int_0^{x/3} \frac{2}{81} x^2 y \, dy \right) dx$$

$$= \int_0^3 \frac{1}{729} x^4 \, dx = \mathbf{\frac{1}{15}}.$$

OR

$$P(X > 3Y) = \int_0^1 \left(\int_{3y}^3 \frac{2}{81} x^2 y \, dx \right) dy = \dots = \mathbf{\frac{1}{15}}.$$

c) Find $P(X+Y>3)$.



$$P(X+Y>3) = \int_0^3 \left(\int_{3-x}^3 \frac{2}{81} x^2 \, y \, dy \right) dx$$

$$= \int_0^3 \frac{1}{81} x^2 \left[9 - (3-x)^2 \right] dx$$

$$= \frac{1}{81} \cdot \int_0^3 \left(6x^3 - x^4 \right) dx$$

$$= \frac{1}{81} \cdot \left(\frac{3}{2} x^4 - \frac{1}{5} x^5 \right) \Big|_0^3 = \frac{1}{81} \cdot \left(\frac{243}{2} - \frac{243}{5} \right) = \mathbf{0.90}.$$

OR

$$P(X+Y>3) = 1 - \int_0^3 \left(\int_0^{3-x} \frac{2}{81} x^2 \, y \, dy \right) dx = 1 - \int_0^3 \frac{1}{81} x^2 (3-x)^2 \, dx$$

$$= 1 - \frac{1}{81} \cdot \int_0^3 \left(9x^2 - 6x^3 + x^4 \right) dx = 1 - \frac{1}{81} \cdot \left(3x^3 - \frac{3}{2} x^4 + \frac{1}{5} x^5 \right) \Big|_0^3$$

$$= 1 - \frac{1}{81} \cdot \left(81 - \frac{243}{2} + \frac{243}{5} \right) = \mathbf{0.90}.$$

d) Are X and Y independent? If not, find $\operatorname{Cov}(X, Y)$.

$$f_X(x) = \int_0^3 \frac{2}{81} x^2 \, y \, dy = \frac{1}{9} x^2, \quad 0 < x < 3,$$

$$f_Y(y) = \int_0^3 \frac{2}{81} x^2 \, y \, dx = \frac{2}{9} y, \quad 0 < y < 3.$$

$f(x,y) = f_X(x) \cdot f_Y(y)$. $\Rightarrow$ X and Y are **independent**. $\operatorname{Cov}(X, Y) = 0$.

2. Let X denote the number of times a photocopy machine will malfunction: 0, 1, 2, or 3 times, on any given month. Let Y denote the number of times a technician is called on an emergency call. The joint p.m.f. $p(x, y)$ is presented in the table below:

	x				
y	0	1	2	3	$p_Y(y)$
0	0.15	0.30	0.05	0	0.50
1	0.05	0.15	0.05	0.05	0.30
2	0	0.05	0.10	0.05	0.20
$p_X(x)$	0.20	0.50	0.20	0.10	1.00

a) Find the probability $P(Y > X)$.

$P(Y > X) = p(0, 1) + p(1, 2) = 0.05 + 0.05 = \mathbf{0.10}$.

b) Find $p_X(x)$, the marginal p.m.f. of X. ↑

c) Find $p_Y(y)$, the marginal p.m.f. of Y. ↑

d) Are X and Y independent? If not, find $Cov(X, Y)$.

X and Y are **NOT independent**.

$E(X) = 0 \times 0.20 + 1 \times 0.50 + 2 \times 0.20 + 3 \times 0.10 = 1.2.$

$E(Y) = 0 \times 0.50 + 1 \times 0.30 + 2 \times 0.20 = 0.7.$

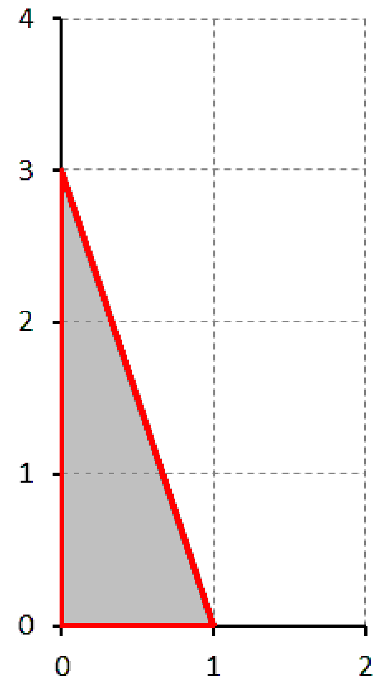
$E(XY) = 1 \times 0.15 + 2 \times 0.05 + 3 \times 0.05 + 2 \times 0.05 + 4 \times 0.10 + 6 \times 0.05 = 1.2.$

$Cov(X, Y) = E(XY) - E(X) \times E(Y) = 1.2 - 1.2 \times 0.70 = \mathbf{0.36}$.

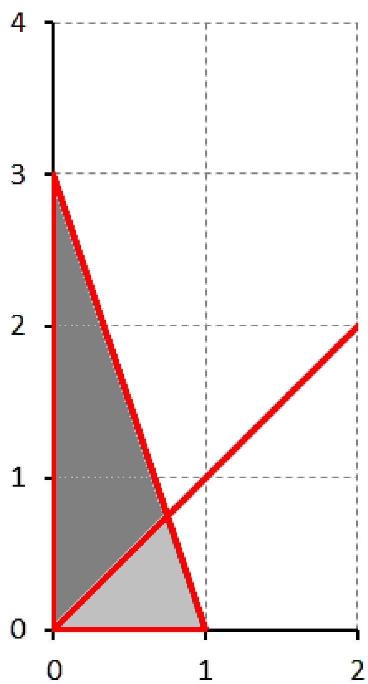
3. Let the joint probability density function for (X, Y) be

$$f(x,y) = \frac{x+y}{2}, \qquad x > 0, \quad y > 0, \quad 3x + y < 3,$$

zero otherwise.



a) Find the probability P (X < Y).



intersection point:

$y = x$ and $x + 3y = 3$

$x = \frac{3}{4}$ and $y = \frac{3}{4}$

$$P(X < Y) = \int_0^{3/4} \left(\int_x^{3-3x} \frac{x+y}{2}\, dy \right) dx$$

$$= \int_0^{3/4} \left(\frac{9}{4} - 3x \right) dx = \mathbf{\frac{27}{32}}.$$

OR

$$P(X < Y) = 1 - \int_0^{3/4} \left(\int_y^{1-(y/3)} \frac{x+y}{2}\, dx \right) dy = 1 - \int_0^{3/4} \left(\frac{1}{4} + \frac{1}{3} y - \frac{8}{9} y^2 \right) dy = \mathbf{\frac{27}{32}}.$$

b) Find the marginal probability density function of X, $f_X(x)$.

$$f_X(x) = \int_0^{3-3x} \frac{x+y}{2}\, dy = \frac{9}{4} - 3x + \frac{3}{4} x^2, \qquad 0 < x < 1.$$

c) Find the marginal probability density function of Y, $f_{\mathrm{Y}}(y)$.

$$f_{\mathrm{Y}}(y) = \int_0^{1-(y/3)} \frac{x+y}{2}\,dx = \frac{1}{4}+\frac{1}{3}y-\frac{5}{36}y^2, \qquad 0 < y < 3.$$

d) Are X and Y independent? If not, find $\mathrm{Cov}(\mathrm{X},\mathrm{Y})$.

The support of (X,Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

OR

$f_{\mathrm{X,Y}}(x,y) \neq f_{\mathrm{X}}(x)\times f_{\mathrm{Y}}(y)$. $\Rightarrow$ X and Y are **NOT independent**.

$$\mathrm{E}(\mathrm{X}) = \int_{-\infty}^{\infty} x\cdot f_{\mathrm{X}}(x)\,dx = \int_0^1 x\cdot\left(\frac{9}{4}-3x+\frac{3}{4}x^2\right)dx$$

$$= \int_0^1 \left(\frac{9}{4}x-3x^2+\frac{3}{4}x^3\right)dx = \frac{9}{8}-1+\frac{3}{16} = \frac{5}{16}.$$

$$\mathrm{E}(\mathrm{Y}) = \int_{-\infty}^{\infty} y\cdot f_{\mathrm{Y}}(y)\,dy = \int_0^3 y\cdot\left(\frac{1}{4}+\frac{1}{3}y-\frac{5}{36}y^2\right)dy$$

$$= \int_0^3 \left(\frac{1}{4}y+\frac{1}{3}y^2-\frac{5}{36}y^3\right)dy = \frac{9}{8}+3-\frac{405}{144} = \frac{21}{16}.$$

$$\mathrm{E}(\mathrm{XY}) = \int_0^1 \left(\int_0^{3-3x} xy\cdot\frac{x+y}{2}\,dy\right)dx = \int_0^1 \left(\frac{x^2}{4}(3-3x)^2+\frac{x}{6}(3-3x)^3\right)dx$$

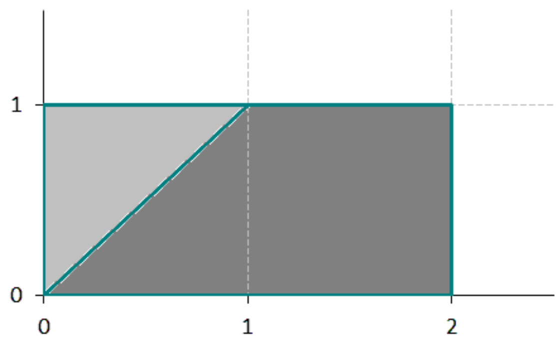
$$= = \int_0^1 \left(\frac{9}{2}x-\frac{45}{4}x^2+9x^3-\frac{9}{4}x^4\right)dx = \frac{9}{4}-\frac{15}{4}+\frac{9}{4}-\frac{9}{20} = \frac{6}{20} = \frac{3}{10}.$$

$$\mathrm{Cov}(\mathrm{X},\mathrm{Y}) = \mathrm{E}(\mathrm{XY})-\mathrm{E}(\mathrm{X})\times\mathrm{E}(\mathrm{Y}) = \frac{3}{10}-\frac{5}{16}\times\frac{21}{16} = -\mathbf{\frac{141}{1280}} \approx -0.11016.$$

4. Let the joint probability density function for (X, Y) be

$$f(x,y) = \frac{x+y}{3}, \qquad 0 < x < 2, \quad 0 < y < 1, \qquad \text{zero otherwise.}$$

a) Find the probability $P(X > Y)$.



$$P(X > Y) = 1 - \int_0^1 \left(\int_0^y \frac{x+y}{3}\, dx \right) dy$$

$$= 1 - \int_0^1 \left(\frac{y^2}{6} + \frac{y^2}{3} \right) dy$$

$$= 1 - \int_0^1 \frac{y^2}{2}\, dy = 1 - \frac{1}{6} = \mathbf{\frac{5}{6}}.$$

OR $$P(X > Y) = \int_0^1 \left(\int_y^2 \frac{x+y}{3}\, dx \right) dy = \ldots$$

OR $$P(X > Y) = \int_0^1 \left(\int_0^x \frac{x+y}{3}\, dy \right) dx + \int_1^2 \left(\int_0^1 \frac{x+y}{3}\, dy \right) dx = \ldots$$

b) Find the marginal probability density function of X, $f_X(x)$.

$$f_X(x) = \int_0^1 \frac{x+y}{3}\, dy = \left(\frac{xy}{3} + \frac{y^2}{6} \right) \Bigg|_0^1 = \frac{2x+1}{6}, \qquad 0 < x < 2.$$

c) Find the marginal probability density function of Y, $f_Y(y)$.

$$f_Y(y) = \int_0^2 \frac{x+y}{3}\, dx = \left(\frac{x^2}{6} + \frac{xy}{3} \right) \Bigg|_0^2 = \frac{2+2y}{3}, \qquad 0 < y < 1.$$

d) Are X and Y independent? If not, find $\text{Cov}(X, Y)$.

Since $f(x,y) \neq f_X(x) \cdot f_Y(y)$, X and Y are **NOT independent**.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_0^2 x \cdot \frac{2x+1}{6}\,dx = \left(\frac{x^3}{9} + \frac{x^2}{12}\right)\Bigg|_0^2 = \frac{11}{9}.$$

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy = \int_0^1 y \cdot \frac{2+2y}{3}\,dy = \left(\frac{y^2}{3} + \frac{y^3}{9}\right)\Bigg|_0^1 = \frac{5}{9}.$$

$$E(XY) = \int_0^2 \left(\int_0^1 x\,y \cdot \frac{x+y}{3}\,dy\right)dx = \int_0^2 \left(\frac{x^2}{6} + \frac{x}{9}\right)dx = \left(\frac{x^3}{18} + \frac{x^2}{18}\right)\Bigg|_0^2 = \frac{2}{3}.$$

$$\text{Cov}(X, Y) = E(XY) - E(X) \times E(Y) = \frac{2}{3} - \frac{11}{9} \cdot \frac{5}{9} = -\mathbf{\frac{1}{81}} \approx -0.012345679.$$

5. Two components of a laptop computer have the following joint probability density function for their useful lifetimes X and Y (in years):

$$f(x,y)=\begin{cases} x\,e^{-x(1+y)} & x\ge 0,\ y\ge 0 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the marginal probability density function of X, $f_X(x)$.

$$f_X(x) = \int_0^\infty x\,e^{-x(1+y)}\,dy = x\,e^{-x}\int_0^\infty e^{-xy}\,dy = e^{-x}, \qquad x\ge 0.$$

b) Find the marginal probability density function of Y, $f_Y(y)$.

$$f_Y(y) = \int_0^\infty x\,e^{-x(1+y)}\,dx = \frac{1}{(1+y)^2}, \qquad y\ge 0.$$

c) What is the probability that the lifetime of at least one component exceeds 1 year (when the manufacturer's warranty expires)?

$$P(X>1 \cup Y>1) = 1 - P(X\le 1 \cap Y\le 1) = 1-\int_0^1\left(\int_0^1 x\,e^{-x(1+y)}\,dy\right)dx$$

$$= 1-\int_0^1 x\,e^{-x}\left(\int_0^1 e^{-xy}\,dy\right)dx = 1-\int_0^1 x\,e^{-x}\left(\frac{1}{x}-\frac{1}{x}e^{-x}\right)dx$$

$$= 1-\int_0^1\left(e^{-x}-e^{-2x}\right)dx = 1-\left(-e^{-x}+\frac{1}{2}e^{-2x}\right)\Big|_0^1$$

$$= 1-\left(-e^{-1}+\frac{1}{2}e^{-2}\right)+\left(-1+\frac{1}{2}\right) = \frac{1}{2}+e^{-1}-\frac{1}{2}e^{-2} \approx 0.800212.$$

OR

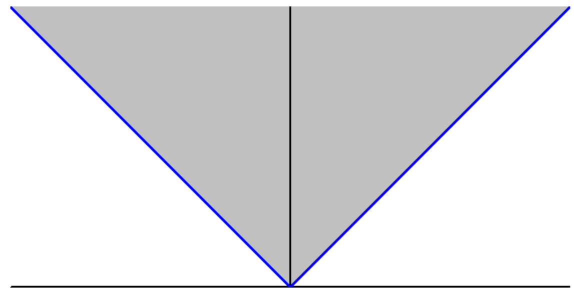
$$P(X>1 \cup Y>1) = P(X>1) + P(Y>1) - P(X>1 \cap Y>1) = \ldots$$

6. Let the joint probability density function for (X, Y) be

$$f(x,y) = \frac{1}{2} e^{-y},$$

$$0 < y < \infty, \qquad -y < x < y,$$

zero otherwise.



a) Find the marginal probability density function of X, $f_X(x)$.

If $x < 0$, $$f_X(x) = \int_{-x}^{\infty} \frac{1}{2} e^{-y}\, dy = \frac{1}{2} e^{x}, \qquad x < 0.$$

If $x > 0$, $$f_X(x) = \int_{x}^{\infty} \frac{1}{2} e^{-y}\, dy = \frac{1}{2} e^{-x}, \qquad x > 0.$$

$$f_X(x) = \frac{1}{2} e^{-|x|}, \qquad -\infty < x < \infty. \qquad \text{(double exponential)}$$

b) Find the marginal probability density function of Y, $f_Y(y)$.

$$f_Y(y) = \int_{-y}^{y} \frac{1}{2} e^{-y}\, dx = y\, e^{-y}, \qquad 0 < y < \infty. \qquad (\text{Gamma},\ \alpha = 2,\ \theta = 1)$$

c) Are X and Y independent? If not, find $\operatorname{Cov}(X, Y)$.

The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

OR

$f_{X,Y}(x,y) \neq f_X(x) \times f_Y(y)$. $\Rightarrow$ X and Y are **NOT independent**.

$E(X) = 0,$ since the distribution of X is symmetric about 0.

$E(Y) = 2,$ since Y has a Gamma distribution, $\alpha = 2,\ \theta = 1$.

$$E(XY) = \int_0^{\infty}\left(\int_{-y}^{y}\frac{1}{2}e^{-y}\,dx\right)dy = \int_0^{\infty}\frac{1}{2}e^{-y}\left(\int_{-y}^{y}dx\right)dy = 0.$$

$$\mathrm{Cov}(X, Y) = E(XY) - E(X) \times E(Y) = \mathbf{0}.$$

Recall:	Independent	$\Rightarrow$	Cov = 0
	Cov = 0	$\not\Rightarrow$	Independent

7. Suppose Jane has a fair 4-sided die, and Dick has a fair 6-sided die. Each day, they roll their dice at the same time (independently) until someone rolls a "1". (Then the person who did not roll a "1" does the dishes.) Find the probability that …

$$p_{\mathrm{J}}(x) = \left(\frac{3}{4}\right)^{x-1} \cdot \left(\frac{1}{4}\right), \quad x = 1, 2, 3, \dots,$$

$$p_{\mathrm{D}}(y) = \left(\frac{5}{6}\right)^{y-1} \cdot \left(\frac{1}{6}\right), \quad y = 1, 2, 3, \dots.$$

a) they roll the first "1" at the same time (after equal number of attempts);

$$\sum_{k=1}^{\infty} p_{\mathrm{J}}(k) \cdot p_{\mathrm{D}}(k) = \sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^{k-1} \cdot \left(\frac{1}{4}\right) \cdot \left(\frac{5}{6}\right)^{k-1} \cdot \left(\frac{1}{6}\right)$$

$$= \left(\frac{1}{24}\right) \cdot \sum_{n=0}^{\infty} \left(\frac{15}{24}\right)^{n} = \left(\frac{1}{24}\right) \cdot \frac{1}{1-\frac{15}{24}} = \mathbf{\frac{1}{9}}.$$

OR

$$(\mathrm{J\,D}) \quad \text{or} \quad (\mathrm{J'D'})(\mathrm{J\,D}) \quad \text{or} \quad (\mathrm{J'D'})(\mathrm{J'D'})(\mathrm{J\,D}) \quad \text{or} \quad \dots$$

$$\left(\frac{1}{4} \cdot \frac{1}{6}\right) + \left(\frac{3}{4} \cdot \frac{5}{6}\right) \cdot \left(\frac{1}{4} \cdot \frac{1}{6}\right) + \left(\frac{3}{4} \cdot \frac{5}{6}\right) \cdot \left(\frac{3}{4} \cdot \frac{5}{6}\right) \cdot \left(\frac{1}{4} \cdot \frac{1}{6}\right) + \dots = \frac{\left(\frac{1}{4} \cdot \frac{1}{6}\right)}{1-\left(\frac{3}{4} \cdot \frac{5}{6}\right)} = \mathbf{\frac{1}{9}}.$$

b) Dick rolls the first "1" before Jane does.

$$\sum_{k=1}^{\infty}\sum_{m=k+1}^{\infty} p_{\mathrm{J}}(m)\cdot p_{\mathrm{D}}(k) = \sum_{k=1}^{\infty}\left(\frac{5}{6}\right)^{k-1}\cdot\left(\frac{1}{6}\right)\cdot\sum_{m=k+1}^{\infty}\left(\frac{3}{4}\right)^{m-1}\cdot\left(\frac{1}{4}\right)$$

$$= \sum_{k=1}^{\infty}\left(\frac{5}{6}\right)^{k-1}\cdot\left(\frac{1}{6}\right)\cdot\left(\frac{3}{4}\right)^{k} = \left(\frac{1}{8}\right)\cdot\sum_{n=0}^{\infty}\left(\frac{15}{24}\right)^{n} = \left(\frac{1}{8}\right)\cdot\frac{1}{1-\frac{15}{24}} = \mathbf{\frac{1}{3}}.$$

OR

$$(\mathrm{J}'\mathrm{D}) \quad \text{or} \quad (\mathrm{J}'\mathrm{D}')(\mathrm{J}'\mathrm{D}) \quad \text{or} \quad (\mathrm{J}'\mathrm{D}')(\mathrm{J}'\mathrm{D}')(\mathrm{J}'\mathrm{D}) \quad \text{or} \quad \ldots$$

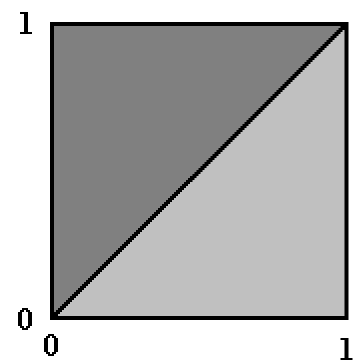
$$\left(\frac{3}{4}\cdot\frac{1}{6}\right) + \left(\frac{3}{4}\cdot\frac{5}{6}\right)\cdot\left(\frac{3}{4}\cdot\frac{1}{6}\right) + \left(\frac{3}{4}\cdot\frac{5}{6}\right)\cdot\left(\frac{3}{4}\cdot\frac{5}{6}\right)\cdot\left(\frac{3}{4}\cdot\frac{1}{6}\right) + \ldots = \frac{\left(\frac{3}{4}\cdot\frac{1}{6}\right)}{1-\left(\frac{3}{4}\cdot\frac{5}{6}\right)} = \mathbf{\frac{1}{3}}.$$

8. Dick and Jane have agreed to meet for lunch between noon (0:00 p.m.) and 1:00 p.m. Denote Jane's arrival time by X, Dick's by Y, and suppose X and Y are independent with probability density functions

$$f_X(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the probability that Jane arrives before Dick. That is, find $P(X < Y)$.

$$f(x, y) = 6x^2 y, \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$$



$$P(X < Y) = \int_0^1 \left(\int_0^y 6x^2 y \, dx \right) dy = \int_0^1 y \left(\int_0^y 6x^2 \, dx \right) dy$$

$$= \int_0^1 y \left(2x^3\right)_0^y dy = \int_0^1 2y^4 \, dy = \left(\frac{2}{5} y^5 \right)_0^1 = \mathbf{\frac{2}{5}}.$$

OR

$$P(X < Y) = \int_0^1 \left(\int_x^1 6x^2 y \, dy \right) dx = \int_0^1 x^2 \left(\int_x^1 6y \, dy \right) dx = \int_0^1 x^2 \left(3y^2\right)_x^1 dx$$

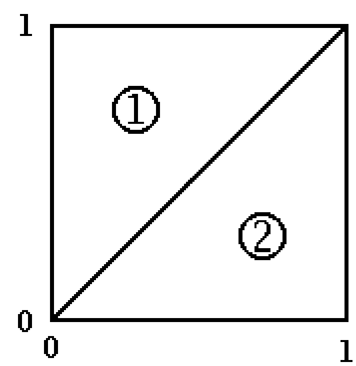
$$= \int_0^1 \left(3x^2 - 3x^4\right) dx = \left(x^3 - \frac{3}{5} x^5 \right)_0^1 = \mathbf{\frac{2}{5}}.$$

b) Find the expected amount of time Jane would have to wait for Dick to arrive.

Hint 1: If Dick arrives first (that is, if X > Y), then Jane's waiting time is zero.
If Jane arrives first (that is, if X < Y), then her waiting time is Y – X.

Hint 2: $E\left(g(X, Y)\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y) \cdot f(x, y)\, dx\, dy$

$f(x, y) = 6x^2 y, \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$



① $y > x$ Jane is waiting for Dick.
Jane's waiting time = $y - x$

② $x > y$ Dick is waiting for Jane.
Jane's waiting time = 0

$$\int_0^1\left(\int_0^y (y-x)\cdot 6x^2 y\, dx\right) dy + \int_0^1\left(\int_0^x 0\cdot 6x^2 y\, dy\right) dx$$

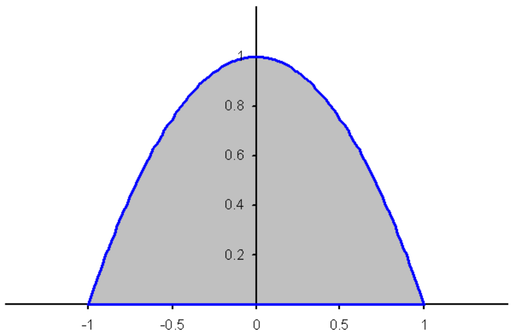
$$= \int_0^1\left(\int_0^y 6x^2y^2\, dx\right) dy - \int_0^1\left(\int_0^y 6x^3 y\, dx\right) dy$$

$$= \int_0^1 2y^5\, dy - \int_0^1 1.5y^5\, dy = \frac{1}{3} - \frac{1}{4} = \mathbf{\frac{1}{12}} \text{ hour} = \mathbf{5} \text{ minutes.}$$

9. Suppose that (X, Y) is uniformly distributed over the region defined by $-1 \le x \le 1$ and $0 \le y \le 1 - x^2$. That is,

$$f(x, y) = C, \quad -1 \le x \le 1,\ 0 \le y \le 1 - x^2, \qquad \text{zero elsewhere.}$$

a) What is the joint probability density function of X and Y? That is, find C.



$$\int_{-1}^{1} \left(\int_{0}^{1-x^2} dy \right) dx = \int_{-1}^{1} \left(1 - x^2\right) dx$$

$$= \left(x - \frac{x^3}{3} \right) \Bigg|_{-1}^{1} = \frac{4}{3}.$$

$$\Rightarrow \quad f_{\text{X,Y}}(x, y) = \mathbf{\frac{3}{4}}, \quad -1 \le x \le 1, \quad 0 \le y \le 1 - x^2.$$

b) Find the marginal probability density function of X, $f_{\text{X}}(x)$.

$$f_{\text{X}}(x) = \int_{0}^{1-x^2} \frac{3}{4}\, dy = \frac{3}{4}\left(1 - x^2\right), \quad -1 \le x \le 1.$$

c) Find the marginal probability density function of Y, $f_{\text{Y}}(y)$.

$$y = 1 - x^2 \qquad\qquad x = \pm\sqrt{1 - y}$$

$$f_{\text{Y}}(y) = \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \frac{3}{4}\, dx = \frac{3}{2}\sqrt{1 - y}, \quad 0 \le y \le 1.$$

10. Let $T_1, T_2, \dots, T_k$ be independent Exponential random variables.

Suppose $E(T_i) = \dfrac{1}{\lambda_i}$, $i = 1, 2, \dots, k$.

That is, $f_{T_i}(t) = \lambda_i e^{-\lambda_i t}$, $t > 0$, $i = 1, 2, \dots, k$.

Denote $T_{\min} = \min(T_1, T_2, \dots, T_k)$.

a) Show that $T_{\min}$ also has an Exponential distribution. What is the mean of $T_{\min}$?

Hint: Consider $P(T_{\min} > t) = P(T_1 > t \text{ AND } T_2 > t \text{ AND } \dots \text{ AND } T_k > t)$.

Since $T_1, T_2, \dots, T_k$ are independent,

$$\begin{aligned} P(T_{\min} > t) &= P(T_1 > t \text{ AND } T_2 > t \text{ AND } \dots \text{ AND } T_k > t) \\ &= P(T_1 > t) \times P(T_2 > t) \times \dots \times P(T_k > t) \\ &= e^{-\lambda_1 t} \times e^{-\lambda_2 t} \times \dots \times e^{-\lambda_k t} \\ &= e^{-(\lambda_1 + \lambda_2 + \dots + \lambda_k)t}, \qquad t > 0. \end{aligned}$$

$$F_{T_{\min}}(t) = P(T_{\min} \le t) = 1 - P(T_{\min} > t) = 1 - e^{-(\lambda_1 + \lambda_2 + \dots + \lambda_k)t}, \qquad t > 0.$$

$$f_{T_{\min}}(t) = (\lambda_1 + \lambda_2 + \dots + \lambda_k)\, e^{-(\lambda_1 + \lambda_2 + \dots + \lambda_k)t}, \qquad t > 0.$$

$\Rightarrow$ $T_{\min}$ has an Exponential distribution with mean $\dfrac{1}{\lambda_1 + \lambda_2 + \dots + \lambda_k}$.

b) Find $P(T_1 = T_{min}) = P(T_1 \text{ is the smallest of } T_1, T_2, \dots, T_k)$
$= P(T_1 < T_2 \text{ AND } \dots \text{ AND } T_1 < T_k)$.

"Hint": A good place to start is to consider T_1, T_2 and show that $P(T_1 < T_2) = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$.

$$P(T_1 < T_2) = \int_0^{\infty} \left(\int_{t_1}^{\infty} \lambda_1 e^{-\lambda_1 t_1} \lambda_2 e^{-\lambda_2 t_2} \, dt_2 \right) dt_1$$

$$= \int_0^{\infty} \lambda_1 e^{-\lambda_1 t_1} e^{-\lambda_2 t_1} \, dt_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Since $T_1, T_2, \dots, T_k$ are independent, their joint probability density function is

$$f(t_1, t_2, \dots, t_k) = \lambda_1 e^{-\lambda_1 t_1} \lambda_2 e^{-\lambda_2 t_2} \dots \lambda_k e^{-\lambda_k t_k},$$

$$t_1 > 0, \quad t_2 > 0, \dots, \quad t_k > 0.$$

$$P(T_1 = T_{min}) = P(T_1 < T_2 \text{ AND } \dots \text{ AND } T_1 < T_k)$$

$$= \int_0^{\infty} \left(\int_{t_1}^{\infty} \dots \int_{t_1}^{\infty} \lambda_1 e^{-\lambda_1 t_1} \lambda_2 e^{-\lambda_2 t_2} \dots \lambda_k e^{-\lambda_k t_k} \, dt_2 \dots dt_k \right) dt_1$$

$$= \int_0^{\infty} \lambda_1 e^{-\lambda_1 t_1} \left(\int_{t_1}^{\infty} \lambda_2 e^{-\lambda_2 t_2} \, dt_2 \right) \dots \left(\int_{t_1}^{\infty} \lambda_k e^{-\lambda_k t_k} \, dt_k \right) dt_1$$

$$= \int_0^{\infty} \lambda_1 e^{-\lambda_1 t_1} e^{-\lambda_2 t_1} \dots e^{-\lambda_k t_1} \, dt_1$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2 + \dots + \lambda_k}.$$