STAT 400
UIUC

Practice Problems #8

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The following are a number of practice problems that may be *helpful* for completing the homework, and will likely be **very useful** for studying for exams.

1. a) Let X and Y be random variables with

$$E(X) = \mu_X = 20, \quad SD(X) = \sigma_X = 7,$$
$$E(Y) = \mu_Y = 8, \quad SD(Y) = \sigma_Y = 2, \quad Corr(X, Y) = \rho = 0.60.$$

Find $E(3X - 5Y)$ and $SD(3X - 5Y)$.

b) Let X and Y be random variables with

$$SD(X) = \sigma_X = 5, \quad SD(Y) = \sigma_Y = 7, \quad SD(3X + 4Y) = 27.$$

Find $Corr(X, Y) = \rho$.

2. Suppose that you wish to invest in two stocks which both have a current price of \$1. The values of these two stocks in one month are described by two random variables, say, X_1 and X_2. Suppose that the expected values and variances of X_1 and X_2 are μ_1, μ_2, σ_1^2, and σ_2^2, respectively. We also assume that the correlation between the stocks is given by ρ.

Let c denote your initial investment, which is to be invested in the stocks, and assume that shares can be bought up to any percentages. Let w denote the percentage of your investment in stock 1. Finally, let P denote the value of your portfolio (investment) after a month. Then we have that $P = c(wX_1 + (1-w)X_2)$, where $0 \le w \le 1$.

a) Find an expression for the expected value of your investment after one month.

b) Find an expression for the variance of your investment after one month.

c) Find the weights that minimize the risk of your investment.
(Hint: in the classical portfolio theory the risk is simply quantified by the variance.)

d) Find the correlation which minimizes the risk of the equally weighted portfolio (i.e., $w = 0.5$).

3. In Anytown, the price of a gallon of milk (X) varies from day to day according to normal distribution with mean \$3.00 and standard deviation \$0.20. The price of a package of Oreo cookies (Y) also varies from day to day according to normal distribution with mean \$2.70 and standard deviation \$0.15. Assume the prices of a gallon of milk and a package of Oreo cookies are independent.

a) Find the probability that on a given day, the price of a package of Oreo cookies is higher than the price of a gallon of milk. That is, find $P(Y > X)$.

b) Alex is planning a Milk-and-Oreos party for his imaginary friends. He buys 4 gallons of milk and 7 packages of Oreo cookies. Find the probability that he paid less than \$30. That is, find $P(4X + 7Y < 30)$.

4. In post-apocalyptic Neverland, clean water is sold in "2-liter" bottles off the back of a truck that arrives every day. The arrival time is normally distributed with mean 3:06 pm and standard deviation 24 minutes. The time it takes for the water to sell out is also normally distributed with mean 30 minutes and standard deviation 7 minutes. The price of a "2-liter" bottle fluctuates from day to day according to a Normal distribution with mean 123 rubles and standard deviation 4 rubles. The amount of water in a "2-liter" bottle varies from bottle to bottle according to a Normal distribution with mean 65 ounces and standard deviation 1.6 ounces. Assume that all days, all times, and all bottles are independent.

a) On April 3, 2020, you buy four "2-liter" bottles of clean water. What is the probability that it costs you more than 500 rubles?

b) On April 3, 2020, you buy four "2-liter" bottles of clean water. What is the probability that you get more than 2 gallons (256 ounces) of clean water?

c) On April 3, 2020, you arrive to the lot where the truck stops at exactly 3:00 pm. What is the probability that the water is sold out by the time you arrive?

d) What is the probability that the truck arrives before 3:00 pm on exactly 3 days in one week?

5. At Sam's Butcher Shop, ground beef packages vary in weight according to a normal distribution with a mean of 3.1 pounds and a standard deviation of 0.2 pounds, and are sold for $1 per pound. Packages of Bratwurst vary in weight according to a normal distribution with a mean of 2.6 pounds and a standard deviation of 0.3 pounds, and are sold for $3 per pound. Anticipating nice weather during the weekend, Dick buys two packages of ground beef and one package of Bratwurst, selecting the packages at random. What is the probability that Dick would exceed the $15 limit "suggested" by his wife Jane? (Assume independence.)

6. The distribution of the baggage weights for passengers using a particular airline has a mean of 20 lbs and a standard deviation of 5 lbs. What is the probability that for (a random sample of) 100 passengers …

a) the total luggage weight is less than 2,100 lbs?

b) the sample mean weight is within 0.5 lb of the overall mean? That is, what is the probability that the sample mean weight is between 19.5 and 20.5 lbs?

7. The amount of cereal dispensed into "16-ounce" boxes of *Captain Crisp* cereal was normally distributed with mean 16.12 ounces and standard deviation 0.20 ounces.

a) What proportion of boxes are "underfilled"? That is, what is the probability that the amount dispensed into a box is less than 16 ounces?

b) Find the probability that exactly 2 out of 9 randomly and independently selected boxes of cereal contain less than 16 ounces.

c) Find the probability that the sample mean amount of cereal for a random sample of 9 boxes is less than 16 ounces.

d) Suppose that the machine can be adjusted to change the mean while the standard deviation remains at 0.20 ounces. What must the mean be so that only 20% of all the boxes are "underfilled"?

8. The weight of an almond varies with mean 0.051 ounce and standard deviation 0.018 ounce.

a) What is the probability (approximately) that the total weight (of a random sample) of 60 almonds is less than 3 ounces?

b) Determine the sample size (the number of almonds) needed to have the probability of at least 0.80 that the total weight is greater than 16 ounces.

9. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $\text{Gamma}(\alpha, \theta)$ distribution. That is,

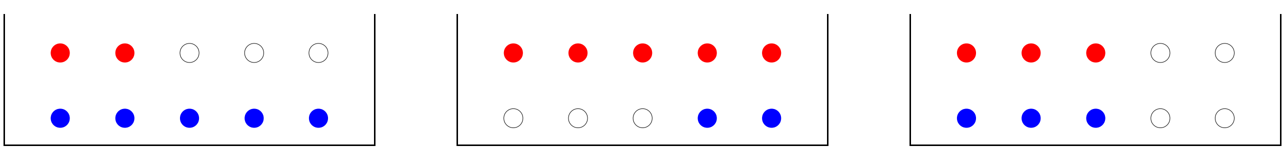
$$f(x; \alpha, \theta) = \frac{1}{\Gamma(\alpha)\,\theta^{\alpha}}\, x^{\alpha-1} e^{-x/\theta}, \qquad 0 < x < \infty, \qquad \alpha > 0,\ \theta > 0.$$

Suppose α is known.

a) Obtain a method of moments estimator of θ, $\tilde{\theta}$.

b) Obtain the maximum likelihood estimator of θ, $\hat{\theta}$.

10. Imagine you are selected as a contestant on *The Price is Right*. The host, Bob Barker, shows you three boxes with marbles in them. Box #1 contains 2 red, 3 white and 5 blue marbles. Box #2 contains 5 red, 3 white and 2 blue marbles. Lastly, Box #3 contains 3 red, 4 white and 3 blue marbles. One of Bob's assistants will pick marbles from one of the boxes after they are hidden behind a curtain. If you correctly guess which box the marbles were picked from, you win a brand new car! Which box (or boxes) would you pick if …



Box #1 Box #2 Box #3

a) one white and one blue marble are selected, with replacement?

b) one red and one blue marble are selected, without replacement?

c) one red, one white, and one blue marble are selected, without replacement?