STAT 400
UIUC

Stepanov
Dalpiaz

Practice Problems #9

SOLUTIONS

The following are a number of practice problems that may be *helpful* for completing the homework, and will likely be **very useful** for studying for exams.

1. (Capture – Recapture) To estimate the populations size of unicorns in Neverland, first N_1 unicorns were captured and tagged. The captured unicorns were then released. One month later, n unicorns were captured. Let X denote the number of tagged unicorns among the ones in the second sample.

a) Construct an estimator for the population size N. Hint: Ask MoM.

X has Hypergeometric distribution. $E(X) = n \cdot \frac{N_1}{N}$.

$$X = n \cdot \frac{N_1}{\tilde{N}}. \quad \Rightarrow \quad \tilde{N} = \frac{n \cdot N_1}{X}.$$

b) Suppose $N_1 = 12$, $n = 10$, and $x = 3$. Obtain $\tilde{N}$, an estimate for the population size N.

$$\tilde{N} = \frac{n \cdot N_1}{x} = \frac{10 \cdot 12}{3} = \mathbf{40}.$$

c) Suppose N = 33, $N_1 = 12$, and $n = 10$. Find the probability that $\tilde{N}$ is within 10 of N. That is, find the probability $P(23 \le \tilde{N} \le 43)$.

x	0	1	2	3	4	5	6	7	8	9	10
$\tilde{N}$	∞	120	60	40	30	24	20	~ 17	15	~ 13	12

$$P(23 \le \tilde{N} \le 43) = P(3 \le X \le 5) = \frac{\binom{12}{3}\binom{21}{7}}{\binom{33}{10}} + \frac{\binom{12}{4}\binom{21}{6}}{\binom{33}{10}} + \frac{\binom{12}{5}\binom{21}{5}}{\binom{33}{10}}$$

$$\approx 0.2764 + 0.2902 + 0.1741 = \mathbf{0.7407}.$$

2. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the distribution with probability density function

$$f(x;\lambda) = \frac{2\sqrt{\lambda}}{\sqrt{\pi}} e^{-\lambda x^2}, \quad x > 0, \qquad \lambda > 0.$$

a) Obtain the maximum likelihood estimator of λ, $\hat{\lambda}$.

$$L(\lambda) = \prod_{i=1}^{n}\left(\frac{2\sqrt{\lambda}}{\sqrt{\pi}} e^{-\lambda x_i^2} \right).$$

$$\ln L(\lambda) = n \cdot \ln 2 + \frac{n}{2} \cdot \ln\lambda - \frac{n}{2} \cdot \ln\pi - \lambda \cdot \sum_{i=1}^{n} x_i^2 .$$

$$(\ln L(\lambda))' = \frac{n}{2\lambda} - \sum_{i=1}^{n} x_i^2 = 0. \qquad \Rightarrow \qquad \hat{\lambda} = \frac{n}{2\sum_{i=1}^{n} X_i^2}.$$

d) Suppose $n = 4$, and $x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 1.1$, $x_4 = 1.7$.

Find the maximum likelihood estimate of λ.

$x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 1.1$, $x_4 = 1.7$.

$$\sum_{i=1}^{n} x_i^2 = 4.5. \qquad \hat{\lambda} = \mathbf{\frac{4}{9}} \approx 0.444.$$

c) Obtain the method of moments estimator of λ, $\tilde{\lambda}$.

$$\mathrm{E}(\mathrm{X}) = \int_0^\infty x \cdot \frac{2\sqrt{\lambda}}{\sqrt{\pi}} e^{-\lambda x^2} dx \qquad u = \lambda x^2 \qquad du = 2\lambda x\, dx$$

$$= \int_0^\infty \frac{1}{\sqrt{\pi\lambda}} e^{-u} du = \frac{1}{\sqrt{\pi\lambda}}.$$

$$\overline{\mathrm{X}} = \frac{1}{\sqrt{\pi\lambda}}. \qquad \Rightarrow \qquad \tilde{\lambda} = \frac{1}{\pi\left(\overline{\mathrm{X}}\right)^2}.$$

d) Suppose $n = 4$, and $x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 1.1$, $x_4 = 1.7$.
Find a method of moments estimate of λ.

$x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 1.1$, $x_4 = 1.7$.

$\bar{x} = 0.9$. $\qquad \tilde{\lambda} \approx 0.393$.

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f_X(x) = f_X(x;\theta) = \left(\theta^2+\theta\right) x^{\theta-1}(1-x), \qquad 0 < x < 1, \qquad \theta > 0.$$

a) Obtain the method of moments estimator of θ, $\tilde{\theta}$.

$$E(X) = \int_0^1 x\cdot\left(\theta^2+\theta\right) x^{\theta-1}(1-x)\,dx = \left(\theta^2+\theta\right)\cdot\int_0^1 \left(x^{\theta}-x^{\theta+1}\right)dx$$

$$= \theta\cdot(\theta+1)\cdot\left(\frac{1}{\theta+1}x^{\theta+1} - \frac{1}{\theta+2}x^{\theta+2}\right)\Bigg|_0^1 = \frac{\theta\cdot(\theta+1)}{(\theta+1)\cdot(\theta+2)} = \frac{\theta}{\theta+2}.$$

OR

Beta distribution, $\alpha = \theta$, $\beta = 2$. $\Rightarrow$ $E(X) = \dfrac{\theta}{\theta+2}$.

$$\frac{\tilde{\theta}}{\tilde{\theta}+2} = \overline{X} \qquad \tilde{\theta} = \overline{X}\cdot\left(\tilde{\theta}+2\right) \qquad \tilde{\theta} - \tilde{\theta}\,\overline{X} = 2\overline{X}$$

$$\Rightarrow \quad \tilde{\theta} = \frac{2\overline{X}}{1-\overline{X}}, \quad \text{where } \overline{X} = \frac{1}{n}\cdot\sum_{i=1}^{n} X_i .$$

b) Is $\tilde{\theta}$ an unbiased estimator of θ? *Justify your answer.*

Consider $g(x) = \dfrac{2x}{1-x}$. Then $g(\overline{X}) = \tilde{\theta}$, $g\left(\dfrac{\theta}{\theta+2}\right) = \theta$.

Also $g''(x) = \dfrac{4}{(1-x)^3} > 0$ for $0 < x < 1$, i.e., $g(x)$ is strictly convex.

By Jensen's Inequality,

$$\mathrm{E}(\tilde{\theta}) = \mathrm{E}[g(\overline{\mathrm{X}})] > g(\mathrm{E}(\overline{\mathrm{X}})) = g(\mu_{\mathrm{X}}) = g\left(\frac{\theta}{\theta+2}\right) = \theta.$$

Therefore, $\tilde{\theta}$ is NOT an unbiased estimator of θ.

c) Suppose $n = 6$, and $x_1 = 0.3$, $x_2 = 0.5$, $x_3 = 0.6$, $x_4 = 0.65$, $x_5 = 0.75$, $x_6 = 0.8$.
Find a method of moments estimate of θ.

$x_1 = 0.3$, $x_2 = 0.5$, $x_3 = 0.6$, $x_4 = 0.65$, $x_5 = 0.75$, $x_6 = 0.8$.

$\bar{x} = 0.6$. $\tilde{\theta} = \mathbf{3}$.

4. Let $\theta > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from a Uniform distribution on interval $(0, \theta)$.

a) Obtain the method of moments estimator of θ, $\tilde{\theta}$.

$$E(X) = \frac{\theta}{2}. \qquad \Rightarrow \qquad \overline{X} = \frac{\tilde{\theta}}{2}. \qquad \Rightarrow \qquad \tilde{\theta} = 2\overline{X}.$$

b) Is $\tilde{\theta}$ an unbiased estimator of θ? *Justify your answer.*

$$E\left(\overline{X}\right) = E(X) = \frac{\theta}{2}. \qquad \Rightarrow \qquad E\left(\tilde{\theta}\right) = E\left(2\overline{X}\right) = \theta. \qquad ✓$$

$\tilde{\theta}$ is unbiased for θ.

c) Find $\operatorname{Var}(\tilde{\theta})$.

$$\tilde{\theta} = 2\overline{X}. \qquad \operatorname{Var}\left(\tilde{\theta}\right) = \operatorname{Var}\left(2\overline{X}\right) = 4\operatorname{Var}\left(\overline{X}\right) = 4\cdot\frac{\sigma^2}{n}.$$

For Uniform $(0, \theta)$, $\quad \sigma^2 = \frac{\theta^2}{12}. \quad \Rightarrow \quad \operatorname{Var}\left(\tilde{\theta}\right) = \frac{\theta^2}{3\cdot n}.$

d) Find $\operatorname{MSE}(\tilde{\theta})$.

$$\operatorname{bias}(\tilde{\theta}) = E(\tilde{\theta}) - \theta = 0 \quad \text{and} \quad \operatorname{Var}(\tilde{\theta}) = \frac{\theta^2}{3\cdot n}.$$

$$\Rightarrow \qquad \operatorname{MSE}(\tilde{\theta}) = E\left[(\tilde{\theta} - \theta)^2\right] = (\operatorname{bias}(\tilde{\theta}))^2 + \operatorname{Var}(\tilde{\theta}) = 0 + \frac{\theta^2}{3\cdot n} = \frac{\theta^2}{3\cdot n}.$$

Note that $\quad \operatorname{MSE}(\tilde{\theta}) = \frac{\theta^2}{3\cdot n} \to 0 \quad$ as $n \to \infty$.

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\theta) = \frac{3+\theta x}{12+8\theta}, \qquad 0 < x < 4, \qquad \theta > -\frac{3}{4}.$$

a) Find the method of moments estimator of θ, $\tilde{\theta}$.

$$\mu = E(X) = \int_0^4 x \cdot \frac{3+\theta x}{12+8\theta}\,dx = \frac{1}{12+8\theta}\cdot\int_0^4 \left(3x+\theta x^2\right)dx$$

$$= \frac{1}{12+8\theta}\cdot\left(\frac{3x^2}{2}+\frac{\theta x^3}{3}\right)\Bigg|_0^4 = \frac{24+\frac{64}{3}\theta}{12+8\theta} = \frac{72+64\theta}{36+24\theta} = \frac{18+16\theta}{9+6\theta}.$$

$$\overline{X} = \frac{18+16\tilde{\theta}}{9+6\tilde{\theta}}. \qquad\qquad 9\overline{X}+6\overline{X}\tilde{\theta} = 18+16\tilde{\theta}.$$

$$\Rightarrow \quad \tilde{\theta} = \frac{12\overline{X}-24}{\frac{64}{3}-8\overline{X}} = \frac{3\overline{X}-6}{\frac{16}{3}-2\overline{X}} = \frac{36\overline{X}-72}{64-24\overline{X}} = \frac{9\overline{X}-18}{16-6\overline{X}}.$$

b) Suppose $n = 5$, and $x_1 = 1.2$, $x_2 = 1.8$, $x_3 = 2.6$, $x_4 = 3.1$, $x_5 = 3.8$.
Find the method of moments estimate of θ.

$x_1 = 1.2$, $x_2 = 1.8$, $x_3 = 2.6$, $x_4 = 3.1$, $x_5 = 3.8$. $\qquad \bar{x} = 2.5.$

$\tilde{\theta} = \mathbf{4.5}.$

c) Suppose $n = 4$, and $x_1 = 1.3$, $x_2 = 2.2$, $x_3 = 3.1$, $x_4 = 3.8$.
Find the method of moments estimate of θ.

$x_1 = 1.3$, $x_2 = 2.2$, $x_3 = 3.1$, $x_4 = 3.8$. $\qquad \bar{x} = 2.6.$

$\tilde{\theta} = \mathbf{13.5}.$

6. **6.4-12** **6.1-12** 6.2-12

Let $X_1, X_2, \ldots, X_n$ be a random sample from Binomial$(1, p)$ (i.e., n Bernoulli trials).

Thus
$$Y = \sum_{i=1}^{n} X_i \quad \text{is Binomial}(n, p).$$

a) Show that $\overline{X} = \dfrac{Y}{n}$ is an unbiased estimator of p.

$$E(Y) = n\,p. \qquad \Rightarrow \qquad E(\overline{X}) = E\left(\frac{Y}{n}\right) = \frac{1}{n}E(Y) = \frac{n\,p}{n} = p.$$

b) Show that $\operatorname{Var}(\overline{X}) = \dfrac{p(1-p)}{n}$.

$$\operatorname{Var}(Y) = n\,p\,(1-p).$$

$$\operatorname{Var}(\overline{X}) = \operatorname{Var}\left(\frac{Y}{n}\right) = \frac{1}{n^2}\operatorname{Var}(Y) = \frac{n\,p(1-p)}{n^2} = \frac{p(1-p)}{n}.$$

c) Show that $E\left[\dfrac{\overline{X}\left(1-\overline{X}\right)}{n}\right] = (n-1)\left[\dfrac{p(1-p)}{n^2}\right]$.

$$E(Y^2) = \operatorname{Var}(Y) + [E(Y)]^2 = n\,p\,(1-p) + n^2p^2.$$

$$E\left(\overline{X}\left(1-\overline{X}\right)\right) = E\left(\frac{Y}{n}\right) - E\left(\frac{Y^2}{n^2}\right) = p - \frac{p(1-p)}{n} - p^2 = (n-1)\left[\frac{p(1-p)}{n}\right].$$

$$\Rightarrow \qquad E\left[\frac{\overline{X}\left(1-\overline{X}\right)}{n}\right] = (n-1)\left[\frac{p(1-p)}{n^2}\right].$$

d) Find the value of c so that $c\,\overline{X}\left(1-\overline{X}\right)$ is an unbiased estimator of $\operatorname{Var}(\overline{X}) = \dfrac{p(1-p)}{n}$.

$$E\left(\overline{X}\left(1-\overline{X}\right)\right) = (n-1)\left[\frac{p(1-p)}{n}\right]. \qquad \Rightarrow \qquad E\left(\frac{1}{(n-1)}\overline{X}\left(1-\overline{X}\right)\right) = \frac{p(1-p)}{n}.$$

$$\Rightarrow \qquad c = \frac{1}{(n-1)}.$$

7 – 8. Let $\lambda > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with the probability density function

$$f(x;\lambda) = 2\lambda^2 x^3 e^{-\lambda x^2}, \qquad x > 0.$$

7. a) Find $E(X^k)$, $k > -4$.

Hint 1: Consider $u = \lambda x^2$ or $u = x^2$.

Hint 2: $\Gamma(a) = \int_0^\infty u^{a-1} e^{-u}\, du$, $a > 0$.

Hint 3: $\Gamma(a) = (a-1)\,\Gamma(a-1)$. Hint 4: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

$$E(X^k) = \int_0^\infty x^k \cdot 2\lambda^2 x^3 e^{-\lambda x^2}\, dx \qquad u = \lambda x^2 \qquad du = 2\lambda x dx$$

$$= \lambda \cdot \int_0^\infty \left(\frac{u}{\lambda}\right)^{\frac{k}{2}+1} e^{-u}\, du = \lambda^{-k/2} \cdot \int_0^\infty u^{\frac{k}{2}+1} e^{-u}\, du = \lambda^{-k/2}\,\Gamma\left(\frac{k}{2}+2\right).$$

OR

$$E(X^k) = \int_0^\infty x^k \cdot 2\lambda^2 x^3 e^{-\lambda x^2}\, dx \qquad u = x^2 \qquad du = 2 x dx$$

$$= \lambda^2 \cdot \int_0^\infty u^{\frac{k}{2}+1} e^{-\lambda u}\, du$$

$$= \lambda^{-k/2}\,\Gamma\left(\frac{k}{2}+2\right) \cdot \int_0^\infty \frac{1}{\Gamma\left(\frac{k}{2}+2\right)} \lambda^{\frac{k}{2}+2} u^{\frac{k}{2}+1} e^{-\lambda u}\, du = \lambda^{-k/2}\,\Gamma\left(\frac{k}{2}+2\right),$$

since $\dfrac{1}{\Gamma\left(\frac{k}{2}+2\right)} \lambda^{\frac{k}{2}+2} u^{\frac{k}{2}+1} e^{-\lambda u}$ is the p.d.f. of Gamma$\left(\alpha = \frac{k}{2}+2,\ \theta = \frac{1}{\lambda}\right)$.

b) Obtain a method of moments estimator of λ, $\tilde{\lambda}$.

$$\mathrm{E}(\mathrm{X}) = \lambda^{-1/2}\,\Gamma\left(\frac{1}{2}+2\right) = \lambda^{-1/2}\cdot\Gamma\left(\frac{5}{2}\right) = \lambda^{-1/2}\cdot\frac{3}{2}\cdot\Gamma\left(\frac{3}{2}\right)$$

$$= \lambda^{-1/2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right) = \lambda^{-1/2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi} = \frac{3}{4}\cdot\sqrt{\frac{\pi}{\lambda}}.$$

$$\frac{3}{4}\cdot\sqrt{\frac{\pi}{\lambda}} = \overline{\mathrm{X}} \qquad \Rightarrow \qquad \tilde{\lambda}_1 = \frac{9\pi}{16\left(\overline{\mathrm{X}}\right)^2}.$$

OR

$$\mathrm{E}(\mathrm{X}^2) = \lambda^{-2/2}\,\Gamma\left(\frac{2}{2}+2\right) = \lambda^{-1}\cdot\Gamma(3) = \lambda^{-1}\cdot 2! = \frac{2}{\lambda}.$$

$$\frac{2}{\lambda} = \overline{\mathrm{X}^2} = \frac{1}{n}\cdot\sum_{i=1}^{n}\mathrm{X}_i^2 \qquad \Rightarrow \qquad \tilde{\lambda}_2 = \frac{2}{\overline{\overline{\mathrm{X}^2}}} = \frac{2n}{\sum_{i=1}^{n}\mathrm{X}_i^2}.$$

c) Suppose $n = 5$, and $x_1 = 0.6$, $x_2 = 1.1$, $x_3 = 2.7$, $x_4 = 3.3$, $x_5 = 4.5$.
Find a method of moments estimate of λ.

$x_1 = 0.6$, $x_2 = 1.1$, $x_3 = 2.7$, $x_4 = 3.3$, $x_5 = 4.5$. $\qquad \overline{x} = \frac{12.2}{5} = 2.44.$

$$\tilde{\lambda}_1 = \frac{9\pi}{16\left(\overline{x}\right)^2} \approx 0.09448\,\pi \approx 0.29682.$$

OR

$x_1 = 0.6$, $x_2 = 1.1$, $x_3 = 2.7$, $x_4 = 3.3$, $x_5 = 4.5$. $\qquad \sum_{i=1}^{n} x_i^2 = 40.$

$$\tilde{\lambda}_2 = \frac{2n}{\sum_{i=1}^{n} x_i^2} = 0.25.$$

8. d) Obtain the maximum likelihood estimator of λ, $\hat{\lambda}$.

$$L(\lambda) = \prod_{i=1}^{n}\left(2\lambda^2 x_i^3 e^{-\lambda x_i^2} \right).$$

$$\ln L(\lambda) = n \cdot \ln 2 + 2n \cdot \ln\lambda + \sum_{i=1}^{n} \ln\left(x_i^3\right) - \lambda \cdot \sum_{i=1}^{n} x_i^2 .$$

$$\left(\ln L(\lambda)\right)' = \frac{2n}{\lambda} - \sum_{i=1}^{n} x_i^2 = 0. \qquad \Rightarrow \qquad \hat{\lambda} = \frac{2n}{\sum_{i=1}^{n} X_i^2}.$$

e) Suppose $n = 5$, and $x_1 = 0.6$, $x_2 = 1.1$, $x_3 = 2.7$, $x_4 = 3.3$, $x_5 = 4.5$.
Find the maximum likelihood estimate of λ.

$x_1 = 0.6$, $x_2 = 1.1$, $x_3 = 2.7$, $x_4 = 3.3$, $x_5 = 4.5$. $\qquad \sum_{i=1}^{n} x_i^2 = 40.$

$$\hat{\lambda} = \frac{2n}{\sum_{i=1}^{n} x_i^2} = \mathbf{0.25}.$$

f) Suppose $n = 5$, and $x_1 = 0.5$, $x_2 = 1.2$, $x_3 = 0.4$, $x_4 = 0.8$, $x_5 = 0.1$.
Find the maximum likelihood estimate of λ.

$x_1 = 0.5$, $x_2 = 1.2$, $x_3 = 0.4$, $x_4 = 0.8$, $x_5 = 0.1$. $\qquad \sum_{i=1}^{n} x_i^2 = 2.5.$

$$\hat{\lambda} = \frac{2n}{\sum_{i=1}^{n} x_i^2} = \mathbf{4}.$$

9. A random sample of size $n = 16$ from $N(\mu, \sigma^2 = 64)$ yielded $\overline{x} = 85$.
Construct the following confidence intervals for μ:

$\overline{x} = 85$ $\sigma = 8$ $n = 16$

σ is known. The confidence interval : $\overline{X} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$.

a) 95%.

$\alpha = 0.05$ $\alpha/2 = 0.025$. $z_{\alpha/2} = 1.96$.

$85 \pm 1.96 \cdot \frac{8}{\sqrt{16}}$ **85 ± 3.92** **(81.08 ; 88.92)**

b) 90%.

$\alpha = 0.10$ $\alpha/2 = 0.05$. $z_{\alpha/2} = 1.645$.

$85 \pm 1.645 \cdot \frac{8}{\sqrt{16}}$ **85 ± 3.29** **(81.71 ; 88.29)**

c) 80%.

$\alpha = 0.20$ $\alpha/2 = 0.10$. $z_{\alpha/2} = 1.28$.

$85 \pm 1.28 \cdot \frac{8}{\sqrt{16}}$ **85 ± 2.56** **(82.44 ; 87.56)**

OR

$\alpha = 0.20$ $\alpha/2 = 0.10$. $z_{\alpha/2} = 1.282$.

$85 \pm 1.282 \cdot \frac{8}{\sqrt{16}}$ **85 ± 2.564** **(82.436 ; 87.564)**

10. What is the minimum sample size required for estimating μ for $N(\mu, \sigma^2 = 64)$ to within ± 3 with confidence level

$\varepsilon = 10$, $\sigma = 8$.

$$n = \left(\frac{z_{\alpha/2} \cdot \sigma}{\varepsilon}\right)^2 = \left(\frac{z_{\alpha/2} \cdot 8}{3}\right)^2.$$

a) 95%. $\alpha = 0.05$ $\alpha/2 = 0.025.$ $z_{\alpha/2} = 1.96.$

$$n = \left(\frac{z_{\alpha/2} \cdot \sigma}{\varepsilon}\right)^2 = \left(\frac{1.96 \cdot 8}{3}\right)^2 \approx 27.318.$$

Round **up**. $n =$ **28**.

b) 90%. $\alpha = 0.10$ $\alpha/2 = 0.05.$ $z_{\alpha/2} = 1.645.$

$$n = \left(\frac{z_{\alpha/2} \cdot \sigma}{\varepsilon}\right)^2 = \left(\frac{1.645 \cdot 8}{3}\right)^2 \approx 19.243.$$

Round **up**. $n =$ **20**.

c) 80%. $\alpha = 0.20$ $\alpha/2 = 0.10.$ $z_{\alpha/2} = 1.28.$

$$n = \left(\frac{z_{\alpha/2} \cdot \sigma}{\varepsilon}\right)^2 = \left(\frac{1.28 \cdot 8}{3}\right)^2 \approx 11.651.$$

Round **up**. $n =$ **12**.

OR

$\alpha = 0.20$ $\alpha/2 = 0.10.$ $z_{\alpha/2} = 1.282.$

$$n = \left(\frac{z_{\alpha/2} \cdot \sigma}{\varepsilon}\right)^2 = \left(\frac{1.282 \cdot 8}{3}\right)^2 \approx 11.687.$$

Round **up**. $n =$ **12**.