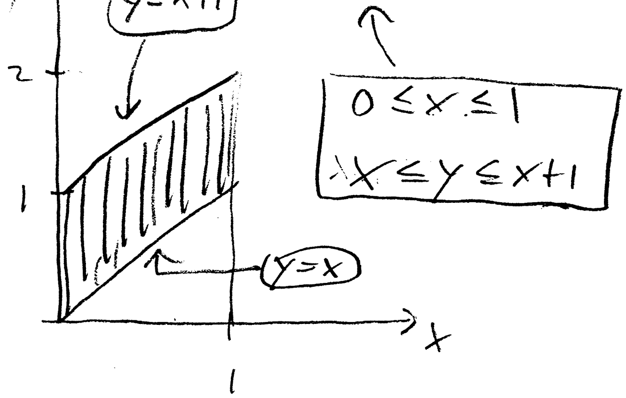
Double integrals

Practice problems — Solutions

1. Set up a double integral of $f(x, y)$ over the region given by $0 < x < 1, x < y < x + 1$.

   Solution:

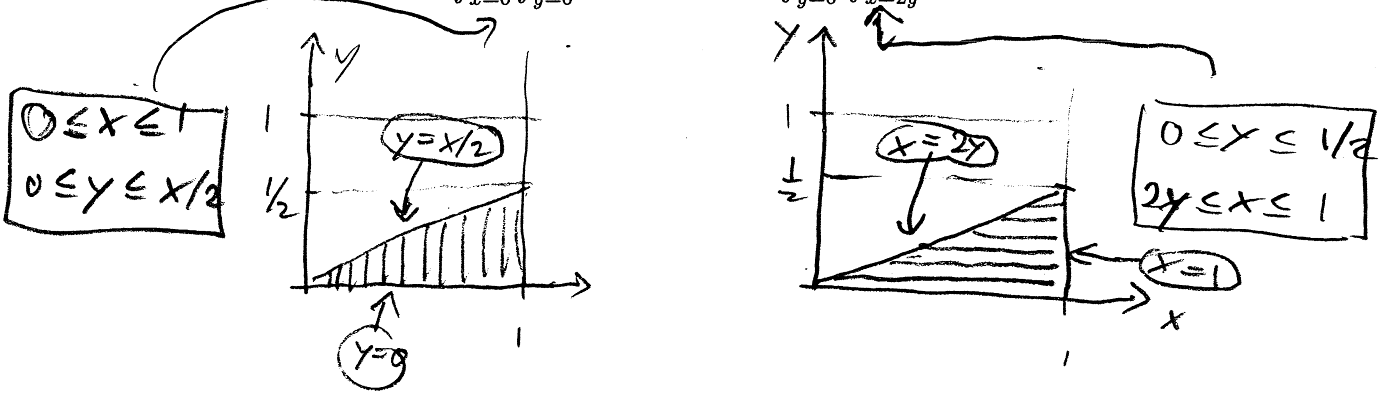
$$\int_{x=0}^{1}\int_{y=x}^{x+1} f(x,y)\,dy\,dx$$



2. Set up a double integral of $f(x, y)$ over the part of the unit square $0 \le x \le 1, 0 \le y \le 1$, on which $y \le x/2$.

   Solution:

$$\int_{x=0}^{1}\int_{y=0}^{x/2} f(x,y)\,dy\,dx \text{ or } \int_{y=0}^{1/2}\int_{x=2y}^{1} f(x,y)\,dx\,dy$$

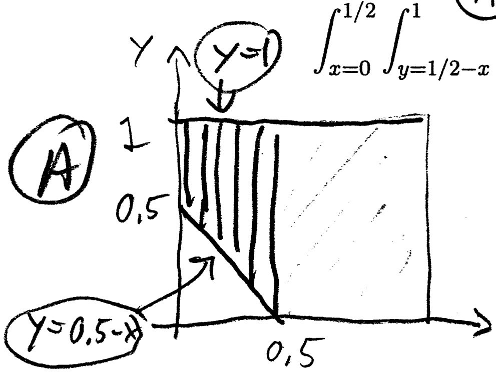


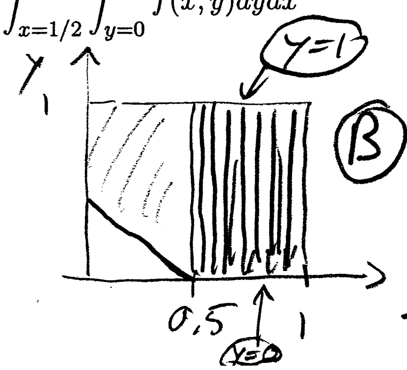
3. Set up a double integral of $f(x, y)$ over the part of the unit square on which $x + y > 0.5$.

   Solution:

A B

$$\int_{x=0}^{1/2}\int_{y=1/2-x}^{1} f(x,y)\,dy\,dx + \int_{x=1/2}^{1}\int_{y=0}^{1} f(x,y)\,dy\,dx$$



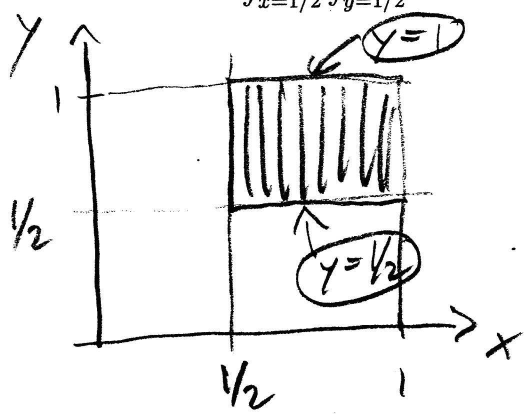


4. Set up a double integral of $f(x,y)$ over the part of the unit square on which **both** x and y are greater than 0.5.

   Solution:

$$\int_{x=1/2}^{1}\int_{y=1/2}^{1} f(x,y)\,dy\,dx$$

$$\boxed{\begin{matrix} 1/2 \le x \le 1 \\ 1/2 \le y \le 1 \end{matrix}}$$



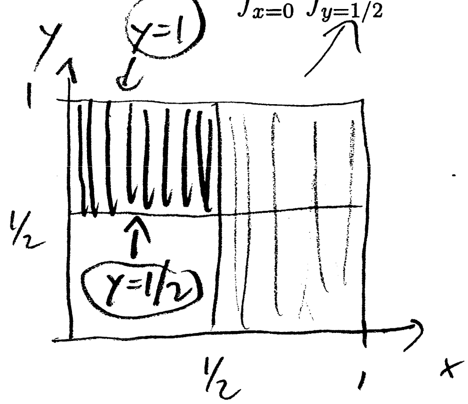
5. Set up a double integral of $f(x,y)$ over the part of the unit square on which **at least one of** x and y is greater than 0.5.

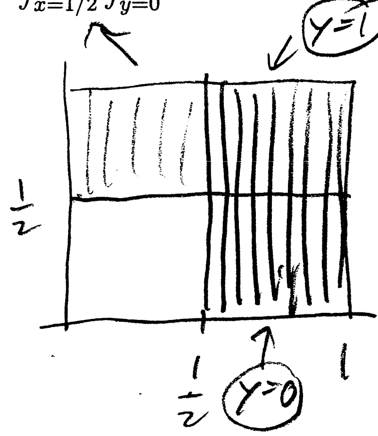
   Solution:

$$\underbrace{\int_{x=0}^{1/2}\int_{y=1/2}^{1} f(x,y)\,dy\,dx}_{A} + \underbrace{\int_{x=1/2}^{1}\int_{y=0}^{1} f(x,y)\,dy\,dx}_{B}$$

A

$$\boxed{\begin{matrix} 0 \le x \le 1/2 \\ 1/2 \le y \le 1 \end{matrix}}$$





B

$$\boxed{\begin{matrix} 1/2 \le x \le 1 \\ 0 \le y \le 1 \end{matrix}}$$

6. Set up a double integral of $f(x,y)$ over the part of the region given by $0 < x < 50 - y < 50$ on which **both** x and y are greater than 20.

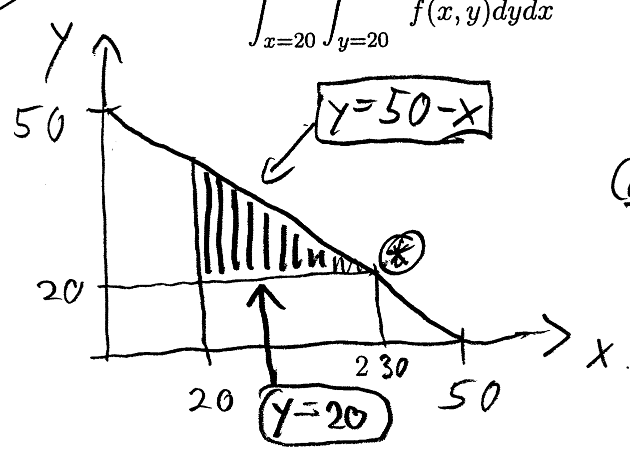
$$\Updownarrow$$

$$\boxed{y < 50 - x}$$

   Solution:

$$\int_{x=20}^{30}\int_{y=20}^{50-x} f(x,y)\,dy\,dx$$

$$\boxed{\begin{matrix} 20 \le x \le 30 \\ 20 \le y \le 50-x \end{matrix}}$$



Calculation for ⊛:

$y = 20$, $x + y = 50$

$\Rightarrow x = 30$

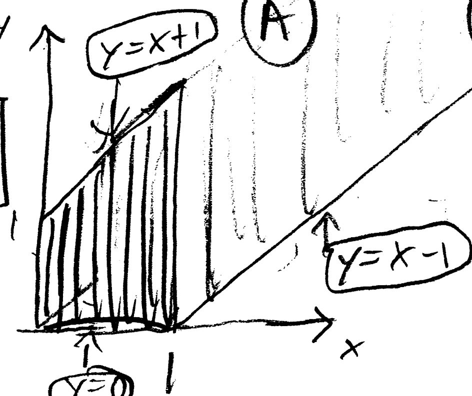
7. Set up a double integral of $f(x, y)$ over the set of all points (x, y) in the first quadrant with $|x - y| \le 1$. $\iff$ $x - 1 \le y \le x + 1$

Solution:

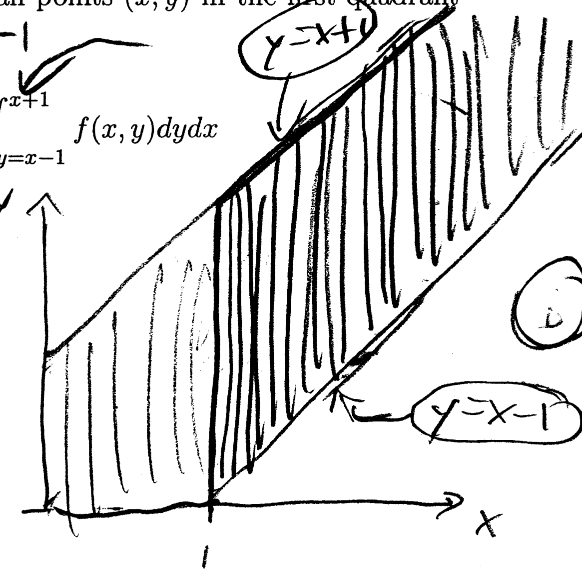
$$\underbrace{\int_{x=0}^{1}\int_{y=0}^{x+1} f(x,y)\,dydx}_{A} + \underbrace{\int_{x=1}^{\infty}\int_{y=x-1}^{x+1} f(x,y)\,dydx}_{B}$$

$\boxed{\begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le x+1 \end{array}}$

A



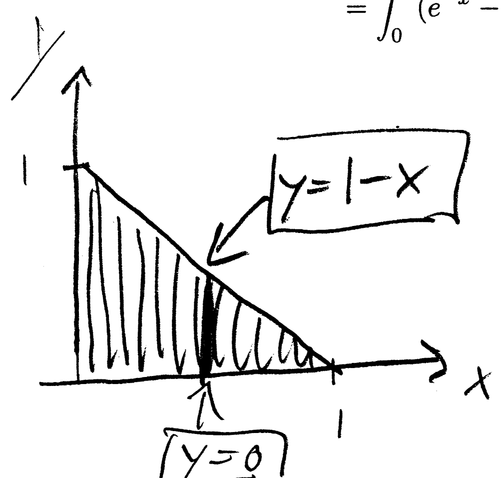
B



8. Evaluate $\iint_R e^{-x-y}dxdy$, where R is the region in the first quadrant in which $x + y \le 1$.

Solution:

$$\int_{x=0}^{1}\int_{y=0}^{1-x} e^{-x}e^{-y}dydx = \int_0^1 e^{-x}(1 - e^{-(1-x)})dx$$
$$= \int_0^1 (e^{-x} - e^{-1})dx = 1 - 2e^{-1}.$$

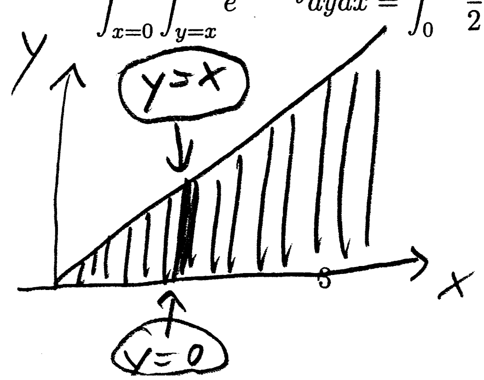


$\boxed{\begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1-x \end{array}}$

9. Evaluate $\iint_R e^{-x-2y}dxdy$, where R is the region in the first quadrant in which $x \le y$

Solution:

$$\int_{x=0}^{\infty}\int_{y=x}^{\infty} e^{-x-2y}dydx = \int_0^{\infty} \frac{1}{2}e^{-3x}dx = \frac{1}{6}$$

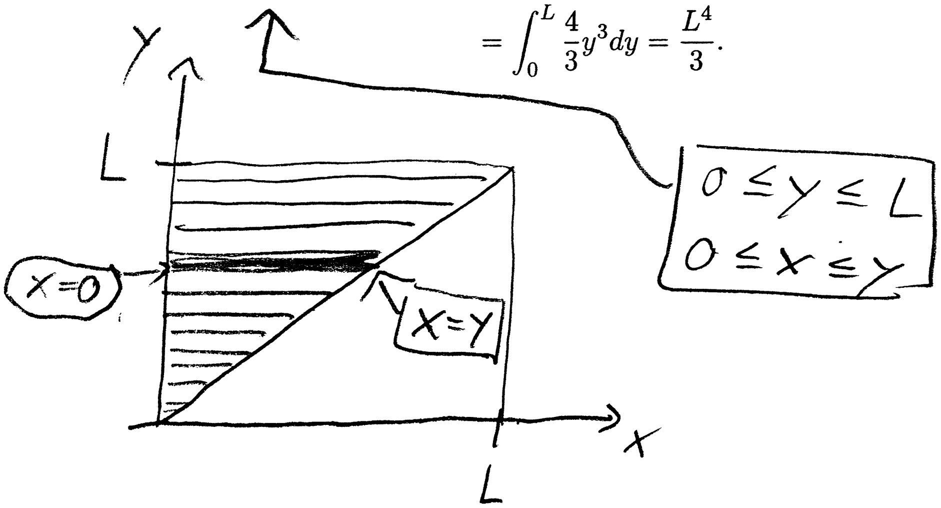


$\boxed{\begin{array}{l} 0 \le x < \infty \\ 0 \le y \le x \end{array}}$

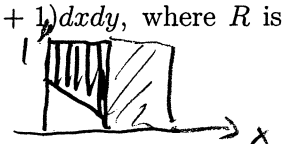
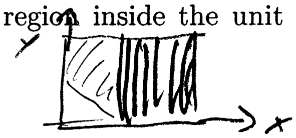
10. Evaluate $\iint_R (x^2+y^2)dxdy$, where R is the region $0 \le x \le y \le L$

Solution:

$$\int_{y=0}^{L}\int_{x=0}^{y}(x^2+y^2)dydx = \int_{y=0}^{L}\left(\frac{1}{3}x^3+y^2x\right)\Big|_{x=0}^{y}dydx$$
$$= \int_0^L \frac{4}{3}y^3dy = \frac{L^4}{3}.$$



11. Evaluate $\iint_R (x-y+1)dxdy$, where R is the region inside the unit square in which $x+y \ge 0.5$.

(Same region as for #3)

Solution:

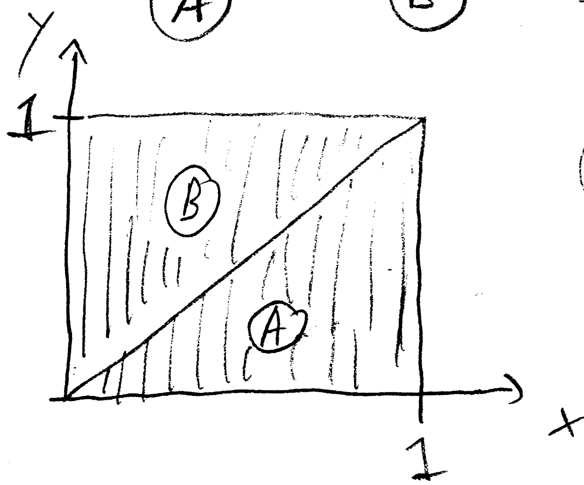
$$\int_{x=0}^{0.5}\int_{y=0.5-x}^{1}(x-y+1)dydx + \int_{x=0.5}^{1}\int_{y=0}^{1}(x-y+1)dydx$$
$$= \int_{x=0}^{0.5}\left(xy-\frac{1}{2}y^2+y\right)\Big|_{y=0.5-x}^{1}dx + \int_{x=0.5}^{1}\left(xy-\frac{1}{2}y^2+y\right)\Big|_{y=0}^{1}dx$$
$$= \int_0^{0.5}\left(x(1-\frac{1}{2}+x)-\frac{1}{2}(1-(\frac{1}{2}-x)^2)+(1-\frac{1}{2}+x)\right)dx$$
$$+\int_{0.5}^{1}\left(x+\frac{1}{2}\right)dx$$
$$= \int_0^{0.5}\left(\frac{1}{8}+x+\frac{3}{2}x^2\right)dx + \left(\frac{1}{2}x^2+\frac{1}{2}x\right)\Big|_{0.5}^{1}$$
$$= \frac{1}{2}\cdot\frac{1}{8}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}\cdot\frac{3}{2}+\frac{3}{8}+\frac{1}{4}=\frac{7}{8}$$

12. Evaluate $\int_0^1 \int_0^1 x \max(x,y) dy dx$.

Solution:

$$\underbrace{\int_{x=0}^{1}\int_{y=0}^{x} x^2 dy dx}_{A} + \underbrace{\int_{x=0}^{1}\int_{y=x}^{1} xy dy dx}_{B} = \int_0^1 \left(x^3 + x\frac{1-x^2}{2} \right) dx$$

$$= \frac{1}{4} + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{8}.$$



A: $\boxed{\begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le x \end{array}}$

B: $\boxed{\begin{array}{l} 0 \le x \le 1 \\ x \le y \le 1 \end{array}}$

(*) In A, $y \le x$, so $\max(x,y) = x$, integrand $= x^2$

In B, $x \le y$, so $\max(x,y) = y$ integrand $= x \cdot y$