STAT 400
UIUC

Answers for 1.1

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1. Suppose a 6-sided die is rolled. The sample space, S, is $\{ 1, 2, 3, 4, 5, 6 \}$.
Consider the following events:

$A = \{$ the outcome is even $\}$,

$B = \{$ the outcome is greater than 3 $\}$,

a) List outcomes in A, B, A', $A \cap B$, $A \cup B$.

$A = \{$ the outcome is even $\} = \{ 2, 4, 6 \}$,

$B = \{$ the outcome is greater than 3 $\} = \{ 4, 5, 6 \}$,

$A' = \{ 1, 3, 5 \}$,

$A \cap B = \{ 4, 6 \}$,

$A \cup B = \{ 2, 4, 5, 6 \}$.

b) Find the probabilities $P(A)$, $P(B)$, $P(A')$, $P(A \cap B)$, $P(A \cup B)$ if the die is balanced (fair).

$P(A) = 3/6$,

$P(B) = 3/6$,

$P(A') = 3/6$,

$P(A \cap B) = 2/6$,

$P(A \cup B) = 4/6$.

c) Suppose the die is loaded so that the probability of an outcome is proportional to the outcome, i.e.

$$P(1) = p, \quad P(2) = 2p, \quad P(3) = 3p, \quad P(4) = 4p, \quad P(5) = 5p, \quad P(6) = 6p.$$

i) Find the value of p that would make this a valid probability model.

$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1.$

$p + 2p + 3p + 4p + 5p + 6p = 21p = 1. \qquad \Rightarrow \qquad p = \mathbf{{}^{1}/_{21}}.$

ii) Find the probabilities $P(A)$, $P(B)$, $P(A')$, $P(A \cap B)$, $P(A \cup B)$.

$P(A) = P(2) + P(4) + P(6) = {}^{2}/_{21} + {}^{4}/_{21} + {}^{6}/_{21} = \mathbf{{}^{12}/_{21}}.$

$P(B) = P(4) + P(5) + P(6) = {}^{4}/_{21} + {}^{5}/_{21} + {}^{6}/_{21} = \mathbf{{}^{15}/_{21}}.$

$P(A') = 1 - P(A) = 1 - {}^{12}/_{21} = \mathbf{{}^{9}/_{21}}.$

OR

$P(A') = P(1) + P(3) + P(5) = {}^{1}/_{21} + {}^{3}/_{21} + {}^{5}/_{21} = \mathbf{{}^{9}/_{21}}.$

$P(A \cap B) = P(4) + P(6) = {}^{4}/_{21} + {}^{6}/_{21} = \mathbf{{}^{10}/_{21}}.$

$P(A \cup B) = P(2) + P(4) + P(5) + P(6) = {}^{2}/_{21} + {}^{4}/_{21} + {}^{5}/_{21} + {}^{6}/_{21} = \mathbf{{}^{17}/_{21}}.$

OR

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = {}^{12}/_{21} + {}^{15}/_{21} - {}^{10}/_{21} = \mathbf{{}^{17}/_{21}}.$

2. Consider a "thick" coin with three possible outcomes of a toss (Heads, Tails, and Edge) for which Heads and Tails are equally likely, but Heads is five times as likely than Edge. What is the probability of Heads ?

$\text{P}(\text{ Heads }) = \text{P}(\text{ Tails }) = p$ for some p. $\text{P}(\text{ Edge }) = \frac{1}{5}\,p$.

$\text{P}(\text{ Heads }) + \text{P}(\text{ Tails }) + \text{P}(\text{ Edge }) = 1.$

$p + p + \frac{1}{5}\,p = 1.$ $\frac{11}{5}\,p = 1.$

$\text{P}(\text{ Heads }) = p = \mathbf{\frac{5}{11}}.$

3. The probability that a randomly selected student at Anytown College owns a bicycle is 0.55, the probability that a student owns a car is 0.30, and the probability that a student owns both is 0.10.

$\text{P}(\text{ B }) = 0.55,$ $\text{P}(\text{ C }) = 0.30,$ $\text{P}(\text{ B} \cap \text{C }) = 0.10.$

a) What is the probability that a student selected at random does not own a bicycle?

$\text{P}(\text{ B}') = 1 - \text{P}(\text{ B }) = 1 - 0.55 = \mathbf{0.45}.$

	C	C′	
B	0.10	0.45	0.55
B′	0.20	0.25	0.45
	0.30	0.70	1.00

b) What is the probability that a student selected at random owns either a car or a bicycle, or both?

$P(B \cup C) = P(B) + P(C) - P(B \cap C) = 0.55 + 0.30 - 0.10 = \mathbf{0.75}$.

OR

$P(B \cup C) = P(B \cap C) + P(B' \cap C) + P(B \cap C') = 0.10 + 0.20 + 0.45 = \mathbf{0.75}$.

OR

$P(B \cup C) = 1 - P(B' \cap C') = 1 - 0.25 = \mathbf{0.75}$.

c) What is the probability that a student selected at random has neither a car nor a bicycle?

$P(B' \cap C') = \mathbf{0.25}$.

4. During the first week of the semester, 80% of customers at a local convenience store bought either beer or potato chips (or both). 60% bought potato chips. 30% of the customers bought both beer and potato chips. What proportion of customers bought beer?

$P(B \cup PC) = 0.80$, $\quad P(PC) = 0.60$, $\quad P(B \cap PC) = 0.30$.

$P(B \cup PC) = P(B) + P(PC) - P(B \cap PC)$.

$0.80 = P(B) + 0.60 - 0.30. \quad \Rightarrow \quad P(B) = \mathbf{0.50}$.

5. Suppose

$P(A) = 0.22,\quad P(B) = 0.25,\quad P(C) = 0.28,$

$P(A \cap B) = 0.11,\quad P(A \cap C) = 0.05,$

$P(B \cap C) = 0.07,\quad P(A \cap B \cap C) = 0.01.$

Find the following:

a) $P(A \cup B)$

b) $P(A' \cap B')$

c) $P(A \cup B \cup C)$

d) $P(A' \cap B' \cap C')$

e) $P(A' \cap B' \cap C)$

f) $P((A' \cap B') \cup C)$

g) $P((A \cup B) \cap C)$

h) $P((B \cap C') \cup A')$

a) $P(A \cup B) = \mathbf{0.36}$.

b) $P(A' \cap B') = \mathbf{0.64}$.

c) $P(A \cup B \cup C) = \mathbf{0.53}$.

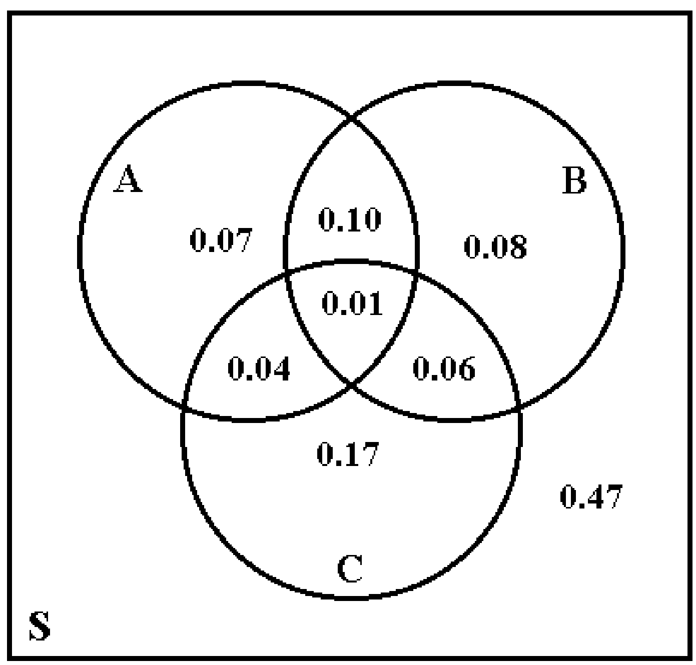
d) $P(A' \cap B' \cap C') = \mathbf{0.47}$.

e) $P(A' \cap B' \cap C) = \mathbf{0.17}$.

f) $P((A' \cap B') \cup C) = \mathbf{0.75}$.

g) $P((A \cup B) \cap C) = \mathbf{0.11}$.

h) $P((B \cap C') \cup A') = \mathbf{0.88}$.



6. Let $a > 2$. Suppose S = $\{ 0, 1, 2, 3, \dots \}$ and

$$P(0) = c, \qquad P(k) = \frac{1}{a^k}, \quad k = 1, 2, 3, \dots .$$

a) Find the value of c (c will depend on a) that makes this is a valid probability distribution.

Must have $\sum\limits_{\text{all } x} p(x) = 1.$ $\quad\Rightarrow\quad$ $c + \sum\limits_{k=1}^{\infty} \frac{1}{a^k} = 1.$

$$\sum_{k=0}^{\infty} b^k = \frac{1}{1-b}, \qquad |b| < 1.$$

$$\sum_{k=1}^{\infty} \frac{1}{a^k} = \left[\sum_{k=0}^{\infty} \frac{1}{a^k} \right] - 1 = \frac{1}{1 - 1/a} - 1 = \frac{1}{a-1}.$$

OR

$$\sum_{k=1}^{\infty} \frac{1}{a^k} = \frac{1}{a} \cdot \sum_{k=0}^{\infty} \frac{1}{a^k} = \frac{1}{a} \cdot \frac{1}{1 - 1/a} = \frac{1}{a-1}.$$

$$c + \frac{1}{a-1} = 1. \qquad c = 1 - \frac{1}{a-1} = \frac{a-2}{a-1} = 2 - \frac{a}{a-1}.$$

b) Find the probability of an odd outcome.

$$\text{P}(\text{odd}) = p(1) + p(3) + p(5) + \dots = \frac{1}{a^1} + \frac{1}{a^3} + \frac{1}{a^5} + \dots$$

$$= \frac{\textit{first term}}{1 - \textit{base}} = \frac{\frac{1}{a}}{1 - \frac{1}{a^2}} = \frac{a}{a^2 - 1}.$$

7. Suppose $S = \{ 0, 1, 2, 3, \dots \}$ and

$$P(0) = p, \qquad P(k) = \frac{1}{2^k \cdot k!}, \quad k = 1, 2, 3, \dots .$$

Find the value of p that would make this a valid probability model.

Must have $\sum_{\text{all } x} p(x) = 1$. $\Rightarrow$ $p + \sum_{k=1}^{\infty} \frac{1}{2^k \cdot k!} = 1$.

Since $\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a$, $\sum_{k=1}^{\infty} \frac{1}{2^k \cdot k!} = \sum_{k=0}^{\infty} \frac{1}{2^k \cdot k!} - 1 = e^{1/2} - 1$.

Therefore, $p + (e^{1/2} - 1) = 1$ and $p = \mathbf{2 - e^{1/2}}$.