Answers for 2.1, 2.2, 2.3

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1. A balanced (fair) coin is tossed twice. Let X denote the number of H's. Construct the probability distribution of X.

S = { HH , HT , TH , TT }
X = 2 1 1 0

Outcomes	x	$f(x)$
TT	0	$1/4$
HT TH	1	$1/2$
HH	2	$1/4$
		1.00

Just for fun:
Suppose P(H) = 0.60,
P(T) = 0.40.

Outcomes	x	$f(x)$
TT	0	0.40 × 0.40 = **0.16**
HT TH	1	0.60 × 0.40 + 0.40 × 0.60 = **0.48**
HH	2	0.60 × 0.60 = **0.36**
		1.00

1½. Suppose Homer Simpson has five coins: 2 nickels, 2 dimes and 1 quarter. Let X denote the amount Bart gets if he steals two coins at random.

a) Construct the probability distribution of X.

Outcomes	x	f(x)	
N N	0.10	$\frac{2}{5} \cdot \frac{1}{4} = \frac{2}{20}$	= 0.1
N D D N	0.15	$\frac{2}{5} \cdot \frac{2}{4} + \frac{2}{5} \cdot \frac{2}{4} = \frac{8}{20}$	= 0.4
D D	0.20	$\frac{2}{5} \cdot \frac{1}{4} = \frac{2}{20}$	= 0.1
N Q Q N	0.30	$\frac{2}{5} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{2}{4} = \frac{4}{20}$	= 0.2
D Q Q D	0.35	$\frac{2}{5} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{2}{4} = \frac{4}{20}$	= 0.2
			1.0

OR

	N_1	N_2	D_1	D_2	Q
N_1	*	0.10	0.15	0.15	0.30
N_2	0.10	*	0.15	0.15	0.30
D_1	0.15	0.15	*	0.20	0.35
D_2	0.15	0.15	0.20	*	0.35
Q	0.30	0.30	0.35	0.35	*

* – do not steal the same coin twice.

x	$f(x)$
0.10	$^2/_{20}$ = 0.1
0.15	$^8/_{20}$ = 0.4
0.20	$^2/_{20}$ = 0.1
0.30	$^4/_{20}$ = 0.2
0.35	$^4/_{20}$ = 0.2
	1.0

OR

Outcomes	x	$f(x)$	
N N	0.10	$\frac{{}_2C_2 \cdot {}_2C_0 \cdot {}_1C_0}{{}_5C_2}$	= 0.1
N D	0.15	$\frac{{}_2C_1 \cdot {}_2C_1 \cdot {}_1C_0}{{}_5C_2}$	= 0.4
D D	0.20	$\frac{{}_2C_0 \cdot {}_2C_2 \cdot {}_1C_0}{{}_5C_2}$	= 0.1
N Q	0.30	$\frac{{}_2C_1 \cdot {}_2C_0 \cdot {}_1C_1}{{}_5C_2}$	= 0.2
D Q	0.35	$\frac{{}_2C_0 \cdot {}_2C_1 \cdot {}_1C_1}{{}_5C_2}$	= 0.2
			1.0

x	$f(x)$	$x \cdot f(x)$	$(x-\mu_X)^2 \cdot f(x)$	$x^2 \cdot f(x)$
0.10	0.1	0.01	0.00144	0.0010
0.15	0.4	0.06	0.00196	0.0090
0.20	0.1	0.02	0.00004	0.0040
0.30	0.2	0.06	0.00128	0.0180
0.35	0.2	0.07	0.00338	0.0245
	1.0	**0.22**	**0.00810**	0.0565

b) Find the expected value of the amount that Bart gets, E(X).

$$\mu_X = E(X) = \sum_{\text{all } x} x \cdot f(x) = \$\mathbf{0.22}.$$

c) Find the standard deviation SD(X).

$$\sigma_X^2 = Var(X) = \sum_{\text{all } x} \left(x - \mu_X \right)^2 \cdot f(x) = \mathbf{0.0081}.$$

OR

$$\sigma_X^2 = Var(X) = \sum_{\text{all } x} x^2 \cdot f(x) - \mu_X^2 = 0.0565 - (0.22)^2 = 0.0565 - 0.0484 = \mathbf{0.0081}.$$

$$\sigma_X = SD(X) = \sqrt{0.0081} = \$\mathbf{0.09}.$$

2. Suppose a random variable X has the following probability distribution:

x	$f(x)$
10	0.20
11	0.40
12	0.30
13	0.10

a) Find the expected value of X, E(X).

x	$f(x)$	$x \cdot f(x)$
10	0.2	2.0
11	0.4	4.4
12	0.3	3.6
13	0.1	1.3
	1.0	11.3

$$\mu_X = E(X) = \sum_{\text{all } x} x \cdot f(x) = \mathbf{11.3}.$$

b) Find the variance of X, Var(X).

x	$f(x)$	$(x-\mu_X)$	$(x-\mu_X)^2 \cdot f(x)$
10	0.2	-1.3	$1.69 \cdot 0.2 = 0.338$
11	0.4	-0.3	$0.09 \cdot 0.4 = 0.036$
12	0.3	0.7	$0.49 \cdot 0.3 = 0.147$
13	0.1	1.7	$2.89 \cdot 0.1 = 0.289$
			0.810

$$\sigma_X^2 = \text{Var(X)} = \sum_{\text{all } x} (x-\mu_X)^2 \cdot f(x) = \mathbf{0.81}.$$

OR

x	$f(x)$	$x^2 \cdot f(x)$
10	0.2	20.0
11	0.4	48.4
12	0.3	43.2
13	0.1	16.9
		128.5

$$\sigma_X^2 = \text{Var(X)} = \sum_{\text{all } x} x^2 \cdot f(x) - [\text{E(X)}]^2 = 128.5 - (11.3)^2 = \mathbf{0.81}.$$

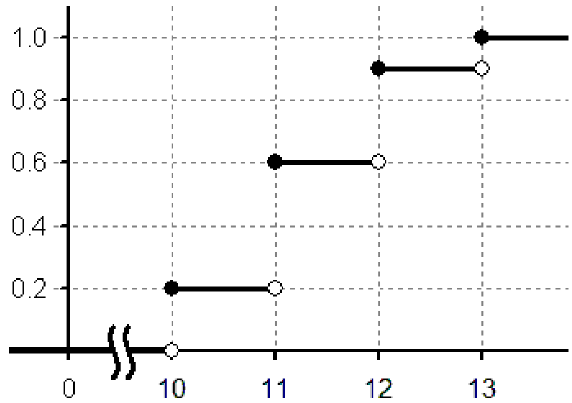
c) Find the standard deviation of X, SD(X).

$$\sigma_X = \text{SD(X)} = \sqrt{\sigma_X^2} = \mathbf{0.9}.$$

d) Find the cumulative distribution function of X, $\text{F}(x) = \text{P}(\text{X} \le x)$.

x	$f(x)$	$\text{F}(x)$
10	0.2	0.2
11	0.4	0.6
12	0.3	0.9
13	0.1	1.0

$$\text{F}(x) = \begin{cases} 0 & x < 10 \\ 0.2 & 10 \le x < 11 \\ 0.6 & 11 \le x < 12 \\ 0.9 & 12 \le x < 13 \\ 1 & x \ge 13 \end{cases}$$



3. Suppose E(X) = 7, SD(X) = 3.

a) Y = 2 X + 3. Find E(Y) and SD(Y).

E(Y) = 2 E(X) + 3 = **17**. SD(Y) = | 2 | SD(X) = **6**.

b) W = 5 – 2 X. Find E(W) and SD(W).

E(W) = 5 – 2 E(X) = **– 9**. SD(Y) = | – 2 | SD(X) = **6**.

3½. Suppose a discrete random variable X has the following probability distribution:

$$f(x) = \left(\frac{1}{2}\right)^x, \qquad x = 1, 2, 3, \dots .$$

a) Verify that this is a valid probability distribution.

- $f(x) \geq 0 \quad \forall x$ ✓
- $\sum_{\text{all } x} f(x) = 1$ $\qquad \sum_{x=1}^{\infty} \left(\frac{1}{2}\right)^x = \frac{\textit{first term}}{1 - \textit{base}} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$ ✓

b) Find P (X is divisible by 3).

$$P(\text{X is divisible by 3}) = P(3) + P(6) + P(9) + P(12) + \dots$$

$$= \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \frac{1}{2^{12}} + \dots = \sum_{k=1}^{\infty} \left(\frac{1}{8}\right)^k = \frac{1}{8} \cdot \frac{1}{1 - \frac{1}{8}} = \mathbf{\frac{1}{7}}.$$

c) Find P (X is divisible by 3 | X is divisible by 2).

$$P(\text{X is divisible by 3} \mid \text{X is divisible by 2})$$

$$= \frac{P(\text{X is divisible by 3} \cap \text{X is divisible by 2})}{P(\text{X is divisible by 2})}$$

$$= \frac{P(\text{X is divisible by 6})}{P(\text{X is divisible by 2})}.$$

$$P(\text{X is divisible by 2}) = P(2) + P(4) + P(6) + P(8) + \dots$$

$$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + \dots = \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k = \frac{1}{4} \cdot \frac{1}{1 - \frac{1}{4}} = \frac{1}{3}.$$

$$P(\text{X is divisible by 6}) = P(6) + P(12) + P(18) + P(24) + \dots$$

$$= \frac{1}{2^6} + \frac{1}{2^{12}} + \frac{1}{2^{18}} + \frac{1}{2^{24}} + \dots = \sum_{k=1}^{\infty} \left(\frac{1}{64}\right)^k = \frac{1}{64} \cdot \frac{1}{1 - \frac{1}{64}} = \frac{1}{63}.$$

$$P(\text{X is divisible by 3} \mid \text{X is divisible by 2})$$

$$= \frac{P(\text{X is divisible by 3} \cap \text{X is divisible by 2})}{P(\text{X is divisible by 2})}$$

$$= \frac{P(\text{X is divisible by 6})}{P(\text{X is divisible by 2})} = \frac{\frac{1}{63}}{\frac{1}{3}} = \mathbf{\frac{1}{21}}.$$

d) Find $\mathrm{E}(\mathrm{X})$.

$$\mathrm{E}(\mathrm{X}) = \sum_{\text{all } x} x \cdot f(x) = \sum_{x=1}^{\infty} x \cdot \left(\frac{1}{2}\right)^{x} = 1 \cdot \frac{1}{2^{1}} + 2 \cdot \frac{1}{2^{2}} + 3 \cdot \frac{1}{2^{3}} + 4 \cdot \frac{1}{2^{4}} + \ldots$$

$$\frac{1}{2}\,\mathrm{E}(\mathrm{X}) = 1 \cdot \frac{1}{2^{2}} + 2 \cdot \frac{1}{2^{3}} + 3 \cdot \frac{1}{2^{4}} + \ldots$$

$$\Rightarrow \qquad \frac{1}{2}\,\mathrm{E}(\mathrm{X}) = \mathrm{E}(\mathrm{X}) - \frac{1}{2}\,\mathrm{E}(\mathrm{X}) = 1 \cdot \frac{1}{2^{1}} + 1 \cdot \frac{1}{2^{2}} + 1 \cdot \frac{1}{2^{3}} + 1 \cdot \frac{1}{2^{4}} + \ldots = 1.$$

$$\Rightarrow \qquad \mathrm{E}(\mathrm{X}) = \mathbf{2}.$$

e) Find the cumulative distribution function of X, $\mathrm{F}(x) = \mathrm{P}(\mathrm{X} \le x)$.

For $k = 1, 2, 3, \ldots$,

$$\mathrm{P}(\mathrm{X} > k) = f(k+1) + f(k+2) + f(k+3) + f(k+4) + \ldots$$

$$= \frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \frac{1}{2^{k+3}} + \frac{1}{2^{k+4}} + \ldots$$

$$= \frac{\textit{first term}}{1 - \textit{base}} = \frac{\frac{1}{2^{k+1}}}{1 - \frac{1}{2}} = \frac{1}{2^{k}}.$$

$$\mathrm{P}(\mathrm{X} > k) = 1 - \mathrm{P}(\mathrm{X} \le k).$$

$$\Rightarrow \qquad \mathrm{P}(\mathrm{X} \le k) = 1 - \frac{1}{2^{k}}.$$

$$\Rightarrow \qquad \mathrm{F}(x) = \mathrm{P}(\mathrm{X} \le x) = \begin{cases} 0 & x < 1 \\ 1 - \dfrac{1}{2^{k}} & k \le x < k+1 \\ & k = 1, 2, 3, \ldots \end{cases}$$

4. Suppose a discrete random variable X has the following probability distribution:

$$P(X = 0) = 2 - \sqrt{e}, \qquad P(X = k) = \frac{1}{2^k \cdot k!}, \quad k = 1, 2, 3, \ldots$$

a) Find $E(X)$.

$$E(X) = \sum_{\text{all } x} x \cdot f(x) = 0 \cdot \left(2 - e^{1/2}\right) + \sum_{k=1}^{\infty} k \cdot \frac{1}{2^k \cdot k!} = \sum_{k=1}^{\infty} \frac{1}{2^k \cdot (k-1)!}$$

$$= \frac{1}{2} \cdot \sum_{k=1}^{\infty} \frac{1}{2^{k-1} \cdot (k-1)!} = \frac{1}{2} \cdot \sum_{n=0}^{\infty} \frac{1}{2^n \cdot n!} = \frac{e^{1/2}}{2}.$$

b) Find $\operatorname{Var}(X)$.

$$E(X(X-1)) = \sum_{k=2}^{\infty} k \cdot (k-1) \cdot \frac{1}{2^k \cdot k!} = \sum_{k=2}^{\infty} \frac{1}{2^k \cdot (k-2)!}$$

$$= \frac{1}{4} \cdot \sum_{k=2}^{\infty} \frac{1}{2^{k-2} \cdot (k-2)!} = \frac{1}{4} \cdot \sum_{n=0}^{\infty} \frac{1}{2^n \cdot n!} = \frac{e^{1/2}}{4}.$$

$$E(X^2) = E(X(X-1)) + E(X) = \frac{3}{4} \cdot e^{1/2}.$$

$$\operatorname{Var}(X) = E(X^2) - [E(X)]^2 = \frac{3}{4} \cdot e^{1/2} - \frac{1}{4} \cdot e.$$