STAT 400
UIUC

Examples for 2.3

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The k^{th} moment of X (the k^{th} moment of X about the origin), μ_k, is given by

$$\mu_k = \mathrm{E}(\mathrm{X}^k) = \sum_{\text{all } x} x^k \cdot f(x)$$

The k^{th} central moment of X (the k^{th} moment of X about the mean), μ_k', is given by

$$\mu_k' = \mathrm{E}((\mathrm{X}-\mu)^k) = \sum_{\text{all } x} (x-\mu)^k \cdot f(x)$$

The moment-generating function of X, $\mathrm{M}_{\mathrm{X}}(t)$, is given by

$$\mathrm{M}_{\mathrm{X}}(t) = \mathrm{E}(e^{t\mathrm{X}}) = \sum_{\text{all } x} e^{tx} \cdot f(x)$$

Theorem 1: $\mathrm{M}_{\mathrm{X}}'(0) = \mathrm{E}(\mathrm{X})$ $\quad$ $\mathrm{M}_{\mathrm{X}}''(0) = \mathrm{E}(\mathrm{X}^2)$

$$\mathrm{M}_{\mathrm{X}}^{(k)}(0) = \mathrm{E}(\mathrm{X}^k)$$

Theorem 2: $\mathrm{M}_{\mathrm{X}_1}(t) = \mathrm{M}_{\mathrm{X}_2}(t)$ for some interval containing 0

$$\Rightarrow \quad f_{\mathrm{X}_1}(x) = f_{\mathrm{X}_2}(x)$$

Theorem 3: Let $\mathrm{Y} = a\mathrm{X} + b$. Then $\mathrm{M}_{\mathrm{Y}}(t) = e^{bt}\mathrm{M}_{\mathrm{X}}(at)$

1. Suppose a random variable X has the following probability distribution:

x	$f(x)$
10	0.20
11	0.40
12	0.30
13	0.10

Find the moment-generating function of X, $\mathrm{M}_{\mathrm{X}}(t)$.

$$\mathrm{M}_{\mathrm{X}}(t) = \mathrm{E}(e^{t\mathrm{X}}) = \sum_{\text{all } x} e^{tx} \cdot f(x)$$

$$= 0.20\,e^{10t} + 0.40\,e^{11t} + 0.30\,e^{12t} + 0.10\,e^{13t}.$$

2. Suppose the moment-generating function of a random variable X is

$$M_X(t) = 0.10 + 0.15\,e^{t} + 0.20\,e^{2t} + 0.25\,e^{-3t} + 0.30\,e^{5t}.$$

Find the expected value of X, E(X).

$$M_X'(t) = 0.15\,e^{t} + 0.40\,e^{2t} - 0.75\,e^{-3t} + 1.50\,e^{5t}.$$

$$E(X) = M_X'(0) = 0.15 + 0.40 - 0.75 + 1.50 = \mathbf{1.30}.$$

OR

x	$f(x)$	$x \cdot f(x)$
0	0.10	0
1	0.15	0.15
2	0.20	0.4
-3	0.25	-0.75
5	0.30	1.5

$$E(X) = \sum_{\text{all } x} x \cdot f(x) = \mathbf{1.30}.$$

3. Suppose a discrete random variable X has the following probability distribution:

$$f(0) = P(X = 0) = 2 - e^{1/2}, \qquad f(k) = P(X = k) = \frac{1}{2^{k} \cdot k!}, \quad k = 1, 2, 3, \ldots$$

a) Find the moment-generating function of X, $M_X(t)$.

$$M_X(t) = \sum_{\text{all } x} e^{tx} \cdot f(x) = 1 \cdot \left(2 - e^{1/2}\right) + \sum_{k=1}^{\infty} e^{tk} \cdot \frac{1}{2^{k} \cdot k!}$$

$$= \left(2 - e^{1/2}\right) + \sum_{k=1}^{\infty} \frac{\left(e^{t}/2\right)^{k}}{k!} = \left(2 - e^{1/2}\right) + \left(e^{e^{t}/2} - 1\right)$$

$$= 1 - e^{1/2} + e^{e^{t}/2}.$$

b) Find the expected value of X, $E(X)$, and the variance of X, $\text{Var}(X)$.

$$M_X'(t) = e^{e^t/2} \cdot e^t/2, \qquad E(X) = M_X'(0) = e^{1/2}/2.$$

$$M_X''(t) = e^{e^t/2} \cdot \left(e^t/2\right)^2 + e^{e^t/2} \cdot e^t/2,$$

$$E(X^2) = M_X''(0) = \frac{3}{4} \cdot e^{1/2}.$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{3}{4} \cdot e^{1/2} - \frac{1}{4} \cdot e.$$

4. Let X be a Binomial(n, p) random variable.
Find the moment-generating function of X.

$$M_X(t) = \sum_{k=0}^{n} e^{tk} \cdot \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \cdot \left(p \cdot e^t\right)^k \cdot (1-p)^{n-k} = [(1-p) + p e^t]^n.$$

5. Let X be a geometric random variable with probability of "success" p.

a) Find the moment-generating function of X.

$$M_X(t) = \sum_{k=1}^{\infty} e^{tk} \cdot (1-p)^{k-1} \cdot p = p \cdot e^t \cdot \sum_{k=1}^{\infty} e^{t(k-1)} \cdot (1-p)^{k-1}$$

$$= p \cdot e^t \cdot \sum_{n=0}^{\infty} \left[(1-p) \cdot e^t\right]^n = \frac{p \cdot e^t}{1-(1-p) \cdot e^t}, \quad t < -\ln(1-p).$$

b) Use the moment-generating function of X to find $E(X)$.

$$M_X'(t) = \frac{p \cdot e^t \cdot \left(1-(1-p)\cdot e^t\right) - p\cdot e^t \cdot \left(-(1-p)\cdot e^t\right)}{\left(1-(1-p)\cdot e^t\right)^2}$$

$$= \frac{p\cdot e^t}{\left(1-(1-p)\cdot e^t\right)^2}, \qquad t < -\ln(1-p).$$

$$E(X) = M_X'(0) = \frac{p}{(p)^2} = \frac{1}{p}.$$

6. a) Find the moment-generating function of a Poisson random variable.

$$M_X(t) = \sum_{k=0}^{\infty} e^{tk} \cdot \frac{\lambda^k \cdot e^{-\lambda}}{k!} = e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\left(\lambda \cdot e^t\right)^k}{k!}$$

$$= e^{-\lambda} \cdot e^{\lambda \cdot e^t} = e^{\lambda \cdot (e^t - 1)}.$$

$$\left(\ln M_X(t)\right)' \Big|_{t=0} = E(X) = \mu_X$$

$$\left(\ln M_X(t)\right)'' \Big|_{t=0} = E(X^2) - [E(X)]^2 = \sigma_X^2$$

b) Find $E(X)$ and $\mathrm{Var}(X)$, where X is a Poisson random variable.

$$\ln M_X(t) = \lambda\,(e^t - 1).$$

$$\left(\ln M_X(t)\right)' = \lambda\, e^t. \qquad \left(\ln M_X(t)\right)' \Big|_{t=0} = E(X) = \lambda.$$

$$\left(\ln M_X(t)\right)'' = \lambda\, e^t. \qquad \left(\ln M_X(t)\right)'' \Big|_{t=0} = \mathrm{Var}(X) = \lambda.$$