STAT 400
UIUC

Answers for 3.2 (Part 2)

Stepanov
Dalpiaz

Gamma Distribution:

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta}, \qquad 0 \le x < \infty$$

$$f(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \qquad 0 \le x < \infty$$

$\mathrm{E}(\mathrm{X}) = \alpha\,\theta$ $\qquad$ $\mathrm{E}(\mathrm{X}) = \alpha/\lambda,$

$\mathrm{Var}(\mathrm{X}) = \alpha\,\theta^2$ $\qquad$ $\mathrm{Var}(\mathrm{X}) = \alpha/\lambda^2$

If T_α has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then

$$\mathrm{F}_{\mathrm{T}_\alpha}(t) = \mathrm{P}(\mathrm{T}_\alpha \le t) = \mathrm{P}(\mathrm{X}_t \ge \alpha),$$

$$\mathrm{P}(\mathrm{T}_\alpha > t) = \mathrm{P}(\mathrm{X}_t \le \alpha - 1),$$

where X_t has a $\text{Poisson}(\lambda t = \frac{t}{\theta})$ distribution.

1. Alex is told that he needs to take bus #5 to the train station. He misunderstands the directions and decides to wait for the fifth bus. Suppose that the buses arrive to the bus stop according to Poisson process with the average rate of one bus per 20 minutes.

X_t = number of buses in t hours. $\qquad$ $\text{Poisson}(\lambda t)$

T_k = arrival time of the kth bus. $\qquad$ Gamma, $\alpha = k$.

one bus per 20 minutes $\quad\Rightarrow\quad$ $\lambda = 3$.

a) Find the probability that Alex would have to wait longer than 1 hour for the fifth bus to arrive.

$$\mathrm{P}(\mathrm{T}_5 > 1) = \mathrm{P}(\mathrm{X}_1 \le 4) = \mathrm{P}(\text{Poisson}(3) \le 4) = \mathbf{0.815}.$$

OR

$$\mathrm{P}(\mathrm{T}_5 > 1) = \int_1^{\infty} \frac{3^5}{\Gamma(5)} t^{5-1} e^{-3t}\,dt = \int_1^{\infty} \frac{3^5}{4!} t^{4} e^{-3t}\,dt = \ldots$$

b) Find the probability that the fifth bus arrives during the second hour.

$$P(1 < T_5 < 2) = P(T_5 > 1) - P(T_5 > 2) = P(X_1 \le 4) - P(X_2 \le 4)$$
$$= P(\text{Poisson}(3) \le 4) - P(\text{Poisson}(6) \le 4) = 0.815 - 0.285 = \mathbf{0.530}.$$

OR

$$P(1 < T_5 < 2) = \int_1^2 \frac{3^5}{\Gamma(5)}\, t^{5-1} e^{-3t}\, dt = \int_1^2 \frac{3^5}{4!}\, t^4 e^{-3t}\, dt = \ldots$$

c) Find the probability that the fifth bus arrives during the third hour.

$$P(2 < T_5 < 3) = P(T_5 > 2) - P(T_5 > 3) = P(X_2 \le 4) - P(X_3 \le 4)$$
$$= P(\text{Poisson}(6) \le 4) - P(\text{Poisson}(9) \le 4) = 0.285 - 0.055 = \mathbf{0.230}.$$

OR

$$P(2 < T_5 < 3) = \int_2^3 \frac{3^5}{\Gamma(5)}\, t^{5-1} e^{-3t}\, dt = \int_2^3 \frac{3^5}{4!}\, t^4 e^{-3t}\, dt = \ldots$$

1.3. Traffic accidents at a particular intersection follow Poisson distribution with an average rate of 1.4 per week.

$\Rightarrow$ 0.2 per day.

a) What is the probability that the next accident will not occur for three days?

Exponential, $\lambda = 0.2$. $\quad P(T > 3) = e^{-0.2 \cdot 3} = e^{-0.6} \approx \mathbf{0.5488}.$

OR

Poisson, 3 days $\Rightarrow$ $\lambda t = 0.2 \cdot 3 = 0.6$. $\quad P(X = 0) = \dfrac{0.6^0 \cdot e^{-0.6}}{0!} \approx \mathbf{0.5488}.$

b) What is the probability that the next accident will occur during the third day? (That is, the time until the next accident is more than two days, but less than three days.)

Exponential, $\lambda = 0.2$.

$$P(2 < T < 3) = P(T > 2) - P(T > 3) = e^{-0.2 \cdot 2} - e^{-0.2 \cdot 3} = e^{-0.4} - e^{-0.6} \approx \mathbf{0.1215}.$$

OR

Poisson, 1 day $\Rightarrow$ $\lambda t = 0.2 \cdot 1 = 0.2$. $\quad P(X = 0) = \dfrac{0.2^{0} \cdot e^{-0.2}}{0!} \approx 0.81873.$

1st day No accident		2nd day No accident		3rd day Accident(s)		
0.81873	×	0.81873	×	0.18127	≈	**0.1215**.

c) What is the probability that the second accident will occur before the end of the third day?

P(the second accident will occur before the end of the third day)

= P(at least 2 accidents will occur in three days)

$$= 1 - \frac{0.6^{0} e^{-0.6}}{0!} - \frac{0.6^{1} e^{-0.6}}{1!} \approx \mathbf{0.1219}.$$

OR

$$\int_0^3 \frac{0.2^{2}}{1} x^{1} e^{-0.2x}\, dx = 0.2^{2} \cdot \left[-\frac{1}{0.2} x e^{-0.2x} - \frac{1}{0.2^{2}} e^{-0.2x} \right] \Bigg|_0^3 \approx \mathbf{0.1219}.$$

d) What is the probability that the fourth accident will occur after the end of the seventh day?

Gamma, $\alpha = 4$, $\lambda = 0.2$. $$P(T_4 > 7 \text{ days}) = \int_7^\infty \frac{0.2^4}{3!} x^3 e^{-0.2x}\, dx = \ldots$$

OR

Gamma, $\alpha = 4$, $\lambda = 1.4$. $$P(T_4 > 1 \text{ week}) = \int_1^\infty \frac{1.4^4}{3!} x^3 e^{-1.4x}\, dx = \ldots$$

OR

$$P(\text{the fourth accident will occur after the end of the seventh day})$$
$$= P(\text{at most 3 accidents will occur in seven days})$$
$$= \frac{1.4^0 e^{-1.4}}{0!} + \frac{1.4^1 e^{-1.4}}{1!} + \frac{1.4^2 e^{-1.4}}{2!} + \frac{1.4^3 e^{-1.4}}{3!} \approx \mathbf{0.9463}.$$

e) What is the probability that the third accident will occur during the fourth day?

$$P(\text{the third accident will occur during the fourth day})$$
$$= P(\text{the third accident will occur after the end of the third day})$$
$$- P(\text{the third accident will occur after the end of the fourth day})$$
$$= P(\text{at most two accidents will occur in three days})$$
$$- P(\text{at most two accidents will occur in four days})$$
$$= \left[\frac{0.6^0 e^{-0.6}}{0!} + \frac{0.6^1 e^{-0.6}}{1!} + \frac{0.6^2 e^{-0.6}}{2!}\right]$$
$$- \left[\frac{0.8^0 e^{-0.8}}{0!} + \frac{0.8^1 e^{-0.8}}{1!} + \frac{0.8^2 e^{-0.8}}{2!}\right] \approx \mathbf{0.0243}.$$

OR

$$P(3 < T_3 < 4) = \int_3^4 \frac{0.2^3}{2} x^2 e^{-0.2x}\, dx = \ldots \approx \mathbf{0.0243}.$$

1.7. During a radio trivia contest, the radio station receives phone calls according to Poisson process with the average rate of five calls per minute.

X_t = number of phone calls in t minutes. Poisson(λt)

T_k = time of the kth phone call. Gamma, $\alpha = k$.

five calls per minute $\Rightarrow$ $\lambda = 5$.

a) Find the probability that we would have to wait less than two minutes for the ninth phone call.

$$P(T_9 < 2) = P(X_2 \geq 9) = 1 - P(X_2 \leq 8) = 1 - P(\text{Poisson}(10) \leq 8) = 1 - 0.333 = \mathbf{0.667}.$$

OR

$$P(T_9 < 2) = \int_0^2 \frac{5^9}{\Gamma(9)}\, t^{9-1} e^{-5t}\, dt = \int_0^2 \frac{5^9}{8!}\, t^8 e^{-5t}\, dt = \ldots$$

b) Find the probability that the ninth phone call would arrive during the third minute.

$$P(2 < T_9 < 3) = P(T_9 > 2) - P(T_9 > 3) = P(X_2 \leq 8) - P(X_3 \leq 8)$$
$$= P(\text{Poisson}(10) \leq 8) - P(\text{Poisson}(15) \leq 8) = 0.333 - 0.037 = \mathbf{0.296}.$$

OR

$$P(2 < T_9 < 3) = \int_2^3 \frac{5^9}{\Gamma(9)}\, t^{9-1} e^{-5t}\, dt = \int_2^3 \frac{5^9}{8!}\, t^8 e^{-5t}\, dt = \ldots$$

2. a) Mistakes that David makes in class occur according to Poisson process with the average rate of one mistake per 10 minutes. Find the probability that the third mistake David makes occurs during the last 15 minutes of a 50-minute class.

Notations: X_t = number of mistakes in t minutes.

T_k = time of the kth mistake.

1 min $\theta = 10, \quad \lambda = \frac{1}{10} = 0.10.$

$$P(35 < T_3 < 50) = P(T_3 > 35) - P(T_3 > 50) = P(X_{35} \le 2) - P(X_{50} \le 2) = \ldots$$

5 min $\theta = 2, \quad \lambda = \frac{1}{2} = 0.50.$

$$P(7 < T_3 < 10) = P(T_3 > 7) - P(T_3 > 10) = P(X_7 \le 2) - P(X_{10} \le 2) = \ldots$$

10 min $\theta = 1, \quad \lambda = 1.$

$$P(3.5 < T_3 < 5) = P(T_3 > 3.5) - P(T_3 > 5) = P(X_{3.5} \le 2) - P(X_5 \le 2) = \ldots$$

$$\ldots = P(\text{Poisson}(3.5) \le 2) - P(\text{Poisson}(5.0) \le 2) = 0.3208 - 0.1247 = \mathbf{0.1961}.$$

OR

1 min $$P(35 < T_3 < 50) = \int_{35}^{50} \frac{0.10^3}{\Gamma(3)} t^{3-1} e^{-0.10t}\, dt = \int_{35}^{50} \frac{1}{2{,}000} t^2 e^{-0.10t}\, dt = \ldots$$

5 min $$P(7 < T_3 < 10) = \int_{7}^{10} \frac{0.50^3}{\Gamma(3)} t^{3-1} e^{-0.50t}\, dt = \int_{7}^{10} \frac{1}{16} t^2 e^{-0.50t}\, dt = \ldots$$

10 min $$P(3.5 < T_3 < 5) = \int_{3.5}^{5.0} \frac{1}{\Gamma(3)} t^{3-1} e^{-t}\, dt = \int_{3.5}^{5.0} \frac{1}{2} t^2 e^{-t}\, dt = \ldots$$

b) Students ask questions in class according to Poisson process with the average rate of one question per 20 minutes. Find the probability that the third question is asked during the last 10 minutes of a 50-minute class.

1 min $\quad \theta = 20, \quad \lambda = \frac{1}{20} = 0.05.$

$$P(40 < T_3 < 50) = P(T_3 > 40) - P(T_3 > 50) = P(X_{40} \le 2) - P(X_{50} \le 2) = \ldots$$

5 min $\quad \theta = 4, \quad \lambda = \frac{1}{4} = 0.25.$

$$P(8 < T_3 < 10) = P(T_3 > 8) - P(T_3 > 10) = P(X_8 \le 2) - P(X_{10} \le 2) = \ldots$$

10 min $\quad \theta = 2, \quad \lambda = \frac{1}{2} = 0.50.$

$$P(4 < T_3 < 5) = P(T_3 > 4) - P(T_3 > 5) = P(X_4 \le 2) - P(X_5 \le 2) = \ldots$$

20 min $\quad \theta = 1, \quad \lambda = 1.$

$$P(2 < T_3 < 2.5) = P(T_3 > 2) - P(T_3 > 2.5) = P(X_2 \le 2) - P(X_{2.5} \le 2) = \ldots$$

$$\ldots = P(\text{Poisson}(2.0) \le 2) - P(\text{Poisson}(2.5) \le 2) = 0.6767 - 0.5438 = \mathbf{0.1329}.$$

OR

1 min $$P(40 < T_3 < 50) = \int_{40}^{50} \frac{0.05^3}{\Gamma(3)} t^{3-1} e^{-0.05t}\, dt = \int_{40}^{50} \frac{1}{16{,}000} t^2 e^{-0.05t}\, dt = \ldots$$

5 min $$P(8 < T_3 < 10) = \int_{8}^{10} \frac{0.25^3}{\Gamma(3)} t^{3-1} e^{-0.25t}\, dt = \int_{8}^{10} \frac{1}{128} t^2 e^{-0.25t}\, dt = \ldots$$

10 min $$P(4 < T_3 < 5) = \int_{4}^{5} \frac{0.50^3}{\Gamma(3)} t^{3-1} e^{-0.50t}\, dt = \int_{4}^{5} \frac{1}{16} t^2 e^{-0.50t}\, dt = \ldots$$

20 min $$P(2 < T_3 < 2.5) = \int_{2.0}^{2.5} \frac{1}{\Gamma(3)} t^{3-1} e^{-t}\, dt = \int_{2.0}^{2.5} \frac{1}{2} t^2 e^{-t}\, dt = \ldots$$

3. Let Y be a random variable with a Gamma distribution with $\alpha = 5$ and $\theta = 3$. Find the probability $P(Y > 18)$ …

a) … by integrating the p.d.f. of the Gamma distribution;

$$P(Y > 18) = \int_{18}^{\infty} \frac{1}{\Gamma(5) \cdot 3^5} \cdot x^{5-1} \cdot e^{-x/3} dx = \int_{18}^{\infty} \frac{1}{5{,}832} \cdot x^4 \cdot e^{-x/3} dx = \ldots$$

b) … by using the relationship between Gamma and Poisson distributions;

$$P(Y > 18) = P(X_{18} \le 4) = \mathbf{0.285} \qquad \text{where } X_{18} \text{ is Poisson}(18/\theta = 6).$$

4. Let T_7 be a random variable with a Gamma distribution with $\alpha = 7$ and $\theta = 5$. Find the probability $P(20 < T_7 < 30)$.

[Text messages arrive according to Poisson process, on average once every 5 minutes. Find the probability that we would have to wait more than 20 minutes but less than 30 minutes for the 7th text message.]

$$P(20 < T_7 < 30) = \int_{20}^{30} \frac{1}{\Gamma(7) \cdot 5^7} \cdot t^{7-1} \cdot e^{-t/5} dt = \int_{20}^{30} \frac{1}{6! \cdot 5^7} \cdot t^6 \cdot e^{-t/5} dt = \ldots$$

OR

$$P(20 < T_7 < 30) = P(T_7 > 20) - P(T_7 > 30) = P(X_{20} \le 6) - P(X_{30} \le 6)$$

$$= P(\text{Poisson}(4) \le 6) - P(\text{Poisson}(6) \le 6) = 0.889 - 0.606 = \mathbf{0.283}.$$

[If the 7th text message arrives after 20 minutes, then we could have received at most 6 text messages during the first 20 minutes. If the average time between the text messages is 5 minutes, then the expected number of text messages in 20 minutes is 4.
If the 7th text message arrives after 30 minutes, then we could have received at most 6 text messages during the first 30 minutes. If the average time between the text messages is 5 minutes, then the expected number of text messages in 30 minutes is 6.]

5. Let Y be a random variable with a Gamma distribution with $\alpha = 5$ and $\theta = 4$ (i.e., $\lambda = 0.25$). Find the probability $P(Y \le 15)$ …

a) … by integrating the p.d.f. of the Gamma distribution;

$$P(Y \le 15) = \int_0^{15} \frac{1}{\Gamma(5) \cdot 4^5} \cdot x^{5-1} \cdot e^{-x/4} dx = \int_0^{15} \frac{1}{24{,}576} \cdot x^4 \cdot e^{-x/4} dx = \ldots$$

b) … by using the relationship between Gamma and Poisson distributions;

$$P(Y \le 15) = P(X_{15} \ge 5) = 1 - P(X_{15} \le 4)$$ where X_{15} is Poisson$(15\lambda = 3.75)$.

EXCEL: $=\text{POISSON}(x, \lambda, 0)$ gives $P(X = x)$

$=\text{POISSON}(x, \lambda, 1)$ gives $P(X \le x)$

	A	B
1	=1-POISSON(4,15/4,1)	
2		

$\Rightarrow$

	A	B
1	0.322452	
2		