STAT 400
UIUC

Answers for 3.2 (Part 1)

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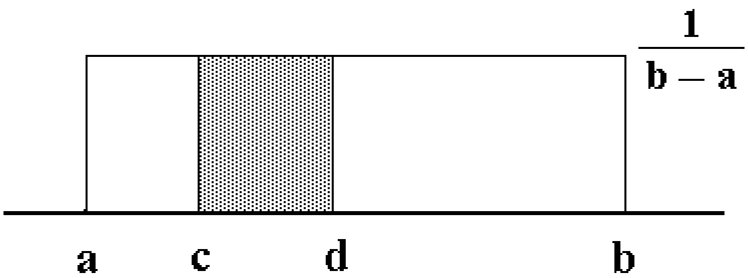
Uniform Distribution over an interval [a , b]:



For Uniform distribution,

$$P(c \le X \le d) = \frac{d-c}{b-a}, \qquad a \le c \le d \le b.$$

$$E(X) = \frac{a+b}{2}, \qquad Var(X) = \frac{(b-a)^2}{12}.$$

0.25. Let X be a random variable distributed uniformly over the interval $[a, b]$. Find the moment-generating function of X.

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\,dx = \int_a^b e^{tx} \cdot \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a} \cdot \left(\frac{e^{tx}}{t}\right)\Bigg|_a^b = \frac{e^{tb} - e^{ta}}{t(b-a)}, \quad t \ne 0.$$

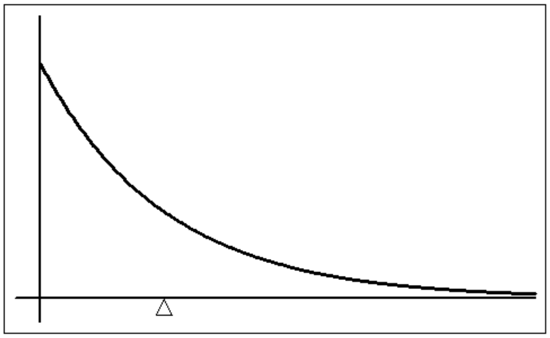
$M_X(0) = 1.$

Exponential Distribution:

$$f(x)=\begin{cases}\frac{1}{\theta}\,e^{-x/\theta} & \text{for } x\ge 0\\ 0 & \text{otherwise}\end{cases}\qquad f(x)=\begin{cases}\lambda\,e^{-\lambda x} & \text{for } x\ge 0\\ 0 & \text{otherwise}\end{cases}$$

$\mathrm{E}(\mathrm{X}) = \theta$,

$\mathrm{Var}(\mathrm{X}) = \theta^2$.



$\mathrm{E}(\mathrm{X}) = {}^1\!/_{\lambda}$,

$\mathrm{Var}(\mathrm{X}) = {}^1\!/_{\lambda^2}$.

0.50. Find the moment generating function of an exponential random variable.

$$M_X(t)=\mathrm{E}(e^{tX})=\int_{-\infty}^{\infty} e^{tx}\cdot f(x)\,dx=\int_0^{\infty} e^{tx}\cdot\frac{1}{\theta}e^{-x/\theta}\,dx$$

$$=\frac{1}{\theta}\cdot\int_0^{\infty} e^{x(t-1/\theta)}\,dx=\frac{1}{\theta}\cdot\left(\frac{e^{x(t-1/\theta)}}{t-1/\theta}\right)\Bigg|_0^{\infty}=\frac{1}{1-t\theta},\qquad t<{}^1\!/_{\theta}.$$

OR

$$M_X(t)=\mathrm{E}(e^{tX})=\int_{-\infty}^{\infty} e^{tx}\cdot f(x)\,dx=\int_0^{\infty} e^{tx}\cdot\lambda e^{-\lambda x}\,dx$$

$$=\lambda\cdot\int_0^{\infty} e^{x(t-\lambda)}\,dx=\lambda\cdot\left(\frac{e^{x(t-\lambda)}}{t-\lambda}\right)\Bigg|_0^{\infty}=\frac{\lambda}{\lambda-t},\qquad t<\lambda.$$

0.75. Suppose X has Exponential distribution with mean θ.

$$\Rightarrow \quad f_X(x) = \frac{1}{\theta} e^{-x/\theta}, \quad x > 0.$$

a) Find $P(X > t)$ for $t > 0$.

$$P(X > t) = \int_t^\infty \frac{1}{\theta} e^{-x/\theta}\, dx = \left(-e^{-x/\theta} \right) \Big|_t^\infty = e^{-t/\theta}, \qquad t > 0.$$

b) Show that for positive t and s,

$$P(X > t + s \mid X > t) = P(X > s)$$

(***memoryless*** property).

$$P(X > t + s \mid X > t) = \frac{P(X > t + s \cap X > t)}{P(X > t)} = \frac{P(X > t + s)}{P(X > t)}$$

$$= \frac{e^{-(t+s)/\theta}}{e^{-t/\theta}} = e^{-s/\theta} = P(X > s).$$

Compare to Geometric distribution:

a) For Geometric (p), $\quad P(X > a) = (1 - p)^a, \quad a = 0, 1, 2, 3, \ldots$.

b) For positive integers a and b,

$$P(X > a + b \mid X > a) = P(X > b)$$

(***memoryless*** property).

1. Suppose the lifetime of a particular brand of light bulbs is exponentially distributed with mean of 400 hours.

Mean = 400 $\Rightarrow$ $\theta = 400$, $\lambda = {}^{1}\!/_{400}$.

a) Find the probability that a randomly selected light bulb would last over 500 hours.

Exponential, $\theta = 400$.

$$P(T > 500) = e^{-500/400} = e^{-1.25} \approx \mathbf{0.2865}.$$

OR

$$P(T > 500) = \int_{500}^{\infty} \frac{1}{400} e^{-x/400}\, dx = \ldots$$

b) Find the probability that 3 out of 7 randomly selected light bulbs would last over 500 hours.

Binomial, $n = 7$, $p = 0.2865$.

$$P(X = 3) = \binom{7}{3} \cdot 0.2865^{3} \cdot 0.7135^{4} \approx \mathbf{0.2133}.$$

c) Find the probability that a randomly selected light bulb would last between 300 hours and 800 hours.

Exponential, $\theta = 400$.

$$P(300 < T < 800) = P(T > 300) - P(T > 800) = e^{-300/400} - e^{-800/400}$$

$$= e^{-0.75} - e^{-2.0} \approx \mathbf{0.337}.$$

OR

$$P(300 < T < 800) \int_{300}^{800} \frac{1}{400} e^{-x/400}\, dx = \ldots$$

2. An insurance policy reimburses a loss up to a benefit limit of C and has a deductible of d. The policyholder's loss, X, follows a distribution with density function $f(x)$. Find the expected value of the benefit paid under the insurance policy?

$$\text{Benefit Paid} = \begin{cases} 0 & x < d \\ x - d & d \le x < C + d \\ C & x \ge C + d \end{cases}$$

$$\text{E}(\text{Benefit Paid}) = \int_0^d 0 \cdot f_X(x)\,dx + \int_d^{C+d} (x-d) \cdot f_X(x)\,dx + \int_{C+d}^{\infty} C \cdot f_X(x)\,dx.$$

For example, if X has an Exponential distribution with mean θ,

$$\text{E}(\text{Benefit Paid}) = \int_d^{C+d} (x-d) \cdot f_X(x)\,dx + \int_{C+d}^{\infty} C \cdot f_X(x)\,dx$$

$$= \int_d^{C+d} (x-d) \cdot \frac{1}{\theta} e^{-x/\theta}\,dx + \int_{C+d}^{\infty} C \cdot \frac{1}{\theta} e^{-x/\theta}\,dx$$

$$= \left(-(x-d) \cdot e^{-x/\theta} - \theta e^{-x/\theta}\right)\Big|_d^{C+d} + \left(-C e^{-x/\theta}\right)\Big|_{C+d}^{\infty}$$

$$= \theta e^{-d/\theta} - \theta e^{-(C+d)/\theta}.$$

For example, if $d = 2$, $C = 10$, X has an Exponential distribution with mean $\theta = 5$,

$$\text{E}(\text{Benefit Paid}) = \int_2^{12} (x-2) \cdot f_X(x)\,dx + \int_{12}^{\infty} 10 \cdot f_X(x)\,dx$$

$$= \int_2^{12} (x-2) \cdot \frac{1}{5} e^{-x/5}\,dx + \int_{12}^{\infty} 10 \cdot \frac{1}{5} e^{-x/5}\,dx$$

$$= \left(-(x-2) \cdot e^{-x/5} - 5 e^{-x/5}\right)\Big|_2^{12} + \left(-10 e^{-x/5}\right)\Big|_{12}^{\infty}$$

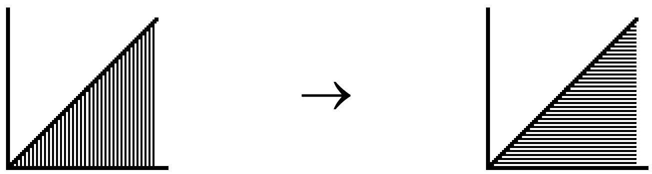
$$= 5 e^{-2/5} - 5 e^{-12/5} = 5 e^{-0.4} - 5 e^{-2.4} \approx \mathbf{2.898}.$$

Fact: Let X be a nonnegative continuous random variable with p.d.f. $f(x)$ and c.d.f. $F(x)$. Then

$$\mathrm{E}(\mathrm{X}) = \int_0^\infty (1 - F(x))\,dx.$$

Proof:

$$\mathrm{E}(\mathrm{X}) = \int_0^\infty x\,f(x)\,dx = \int_0^\infty \left(\int_0^x dy \right) f(x)\,dx = \int_0^\infty \left(\int_0^x f(x)\,dy \right) dx$$



$$\int_0^\infty \left(\int_0^x f(x)\,dy \right) dx = \int_0^\infty \left(\int_y^\infty f(x)\,dx \right) dy$$

$$\Rightarrow \quad \mathrm{E}(\mathrm{X}) = \int_0^\infty \left(\int_y^\infty f(x)\,dx \right) dy = \int_0^\infty \mathrm{P}(\mathrm{X} > y)\,dy = \int_0^\infty (1 - F(y))\,dy.$$

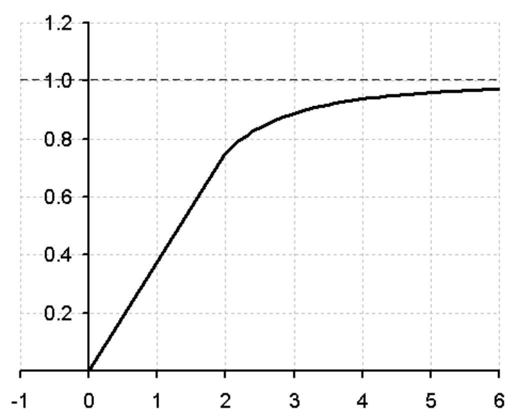
Example: (**2.** from Examples for 3.3)

Let X be a continuous random variable with the cumulative distribution function

$$\mathrm{F}(x) = 0, \qquad x < 0,$$

$$\mathrm{F}(x) = \frac{3}{8} \cdot x, \qquad 0 \le x \le 2,$$

$$\mathrm{F}(x) = 1 - \frac{1}{x^2}, \qquad x > 2.$$



c) $$\mathrm{E}(\mathrm{X}) = \int_0^\infty (1 - \mathrm{F}(x))\,dx = \int_0^2 \left(1 - \frac{3}{8} \cdot x\right) dx + \int_2^\infty \left(\frac{1}{x^2}\right) dx$$

$$= 2 - \frac{3}{8} \cdot \left(\frac{x^2}{2}\right) \Bigg|_0^2 + \left(-\frac{1}{x}\right) \Bigg|_2^\infty = 2 - \frac{3}{4} + \frac{1}{2} = \mathbf{\frac{7}{4}} = \mathbf{1.75}.$$

Example: Find the expected value of an Exponential(θ) distribution.

For Exponential(θ), $\quad 1 - \mathrm{F}(x) = \mathrm{P}(\mathrm{X} > x) = e^{-x/\theta}, \quad t > 0.$

$$\mathrm{E}(\mathrm{X}) = \int_0^{\infty} (1 - F(x))\,dx = \int_0^{\infty} e^{-x/\theta}\,dx = \theta.$$

Fact: Let X be a random variable of the discrete type with pmf $p(x)$ that is positive on the nonnegative integers and is equal to zero elsewhere. Then

$$\mathrm{E}(\mathrm{X}) = \sum_{x=0}^{\infty} [1 - \mathrm{F}(x)],$$

where $\mathrm{F}(x)$ is the cdf of X.

"Proof":

$$1 - \mathrm{F}(x) = \mathrm{P}(\mathrm{X} > x) = p(x+1) + p(x+2) + p(x+3) + p(x+4) + \dots$$

$$\begin{array}{ll}
1 - \mathrm{F}(0) & p(1) + p(2) + p(3) + p(4) + p(5) + p(6) + p(7) + \dots \\
1 - \mathrm{F}(1) & \phantom{p(1) +{}} p(2) + p(3) + p(4) + p(5) + p(6) + p(7) + \dots \\
1 - \mathrm{F}(2) & \phantom{p(1) + p(2) +{}} p(3) + p(4) + p(5) + p(6) + p(7) + \dots \\
1 - \mathrm{F}(3) & \phantom{p(1) + p(2) + p(3) +{}} p(4) + p(5) + p(6) + p(7) + \dots \\
1 - \mathrm{F}(4) & \phantom{p(1) + p(2) + p(3) + p(4) +{}} p(5) + p(6) + p(7) + \dots \\
\quad\dots & \qquad\qquad\dots
\end{array}$$

$$\Rightarrow \quad \sum_{x=0}^{\infty} [1 - \mathrm{F}(x)] = 1 \times p(1) + 2 \times p(2) + 3 \times p(3) + 4 \times p(4) + \dots = \mathrm{E}(\mathrm{X}).$$

Example: Find the expected value of a Geometric(p) distribution.

For Geometric(p), $\quad 1 - \mathrm{F}(x) = \mathrm{P}(\mathrm{X} > x) = (1 - p)^x, \quad x = 0, 1, 2, \dots$.

$$\mathrm{E}(\mathrm{X}) = \sum_{x=0}^{\infty} [1 - \mathrm{F}(x)] = \sum_{x=0}^{\infty} [1 - p]^x = \frac{1}{1 - [1 - p]} = \frac{1}{p}.$$