Answers for 4.1, 4.4 (2)

Stepanov
Dalpiaz

Independent Random Variables

1. Consider the following joint probability distribution $p(x, y)$ of two random variables X and Y:

x \ y	0	1	2	
1	0.15	0.10	0	0.25
2	0.25	0.30	0.20	0.75
	0.40	0.40	0.20	

Recall: A and B are independent if and only if $P(A \cap B) = P(A) \cdot P(B)$.

a) Are events $\{X = 1\}$ and $\{Y = 1\}$ independent?

$$P(X = 1 \cap Y = 1) = p(1, 1) = 0.10 = 0.25 \times 0.40 = P(X = 1) \times P(Y = 1).$$

$\{X = 1\}$ and $\{Y = 1\}$ are **independent**.

Def Random variables X and Y are **independent** if and only if

discrete $\quad p(x, y) = p_X(x) \cdot p_Y(y) \quad$ for all x, y.

continuous $\quad f(x, y) = f_X(x) \cdot f_Y(y) \quad$ for all x, y.

$$F(x, y) = P(X \le x, Y \le y). \qquad f(x, y) = \partial^2 F(x, y) / \partial x \, \partial y.$$

Def Random variables X and Y are **independent** if and only if

$$F(x, y) = F_X(x) \cdot F_Y(y) \quad \text{for all } x, y.$$

b) Are random variables X and Y independent?

$$p(1, 0) = 0.15 \ne 0.25 \times 0.40 = p_X(1) \times p_Y(0).$$

X and Y are **NOT independent**.

2. Let the joint probability density function for (X, Y) be

$$f(x,y) = \begin{cases} 60\,x^2\,y & 0 \le x \le 1,\ 0 \le y \le 1,\ x + y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Recall: $f_X(x) = 30\,x^2\,(1-x)^2, \quad 0 < x < 1,$

$f_Y(y) = 20\,y\,(1-y)^3, \quad 0 < y < 1.$

Are random variables X and Y independent?

The support of (X, Y) is not a rectangle.

X and Y are **NOT independent**.

3. Let the joint probability density function for (X, Y) be

$$f(x,y) = \begin{cases} x + y & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent?

$$\begin{aligned} f_1(x) &= \int_0^1 (x+y)\,dy \\ &= \left[xy + \frac{1}{2}y^2\right]_0^1 = x + \frac{1}{2}, \qquad 0 \le x \le 1 \ ; \\ f_2(y) &= \int_0^1 (x+y)\,dx = y + \frac{1}{2}, \qquad 0 \le y \le 1; \\ f(x,y) &= x + y \ne \left(x + \frac{1}{2}\right)\left(y + \frac{1}{2}\right) = f_1(x)f_2(y). \end{aligned}$$

X and Y are **NOT independent**.

4. Let the joint probability density function for (X, Y) be

$$f(x,y)=\begin{cases} 12\,x(1-x)\,e^{-2y} & 0 \le x \le 1,\ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent?

$$f_X(x) = \int_0^\infty 12\,x(1-x)e^{-2y}\,dy = 6\,x(1-x), \quad 0 < x < 1.$$

$$f_Y(y) = \int_0^1 12\,x(1-x)e^{-2y}\,dx = 2\,e^{-2y}, \quad y > 0.$$

Since $f(x,y) = f_X(x)\cdot f_Y(y)$ for all x, y, X and Y are **independent**.

If random variables X and Y are independent, then

$$E(g(X)\cdot h(Y)) = E(g(X))\cdot E(h(Y)).$$

5. Suppose the probability density functions of T_1 and T_2 are

$$f_{T_1}(x) = \alpha\,e^{-\alpha x}, \quad x > 0, \qquad f_{T_2}(y) = \beta\,e^{-\beta y}, \quad y > 0,$$

respectively. Suppose T_1 and T_2 are independent. Find $P(2T_1 > T_2)$.

$$P(2T_1 > T_2) = \int_0^\infty\left(\int_{y/2}^\infty\left(\alpha\beta\,e^{-\alpha x-\beta y}\right)dx\right)dy = \int_0^\infty \beta\,e^{-\beta y}\left(\int_{y/2}^\infty\left(\alpha\,e^{-\alpha x}\right)dx\right)dy$$

$$= \int_0^\infty \beta\,e^{-\beta y}\left(e^{-\alpha y/2}\right)dy = \int_0^\infty \beta\,e^{-(\alpha/2+\beta)y}\,dy = \frac{\mathbf{2\beta}}{\boldsymbol{\alpha}+\mathbf{2\beta}}.$$

6. Let X and Y be two independent random variables, X has a Geometric distribution with the probability of "success" $p = {}^1\!/_3$, Y has a Poisson distribution with mean 3. That is,

$$p_X(x) = \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right)^{x-1}, \qquad x = 1, 2, 3, \ldots,$$

$$p_Y(y) = \frac{3^y e^{-3}}{y!}, \quad y = 0, 1, 2, 3, \ldots.$$

a) Find $P(X = Y)$.

$$P(X = Y) = \sum_{k=1}^{\infty} p_X(k) \cdot p_Y(k) = \sum_{k=1}^{\infty} \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right)^{k-1} \cdot \frac{3^k e^{-3}}{k!}$$

$$= e^{-3} \cdot \sum_{k=1}^{\infty} \frac{2^{k-1}}{k!} = \frac{e^{-3}}{2} \cdot \left[\sum_{k=0}^{\infty} \frac{2^k}{k!} - 1 \right] = \frac{e^{-3}}{2} \cdot \left[e^2 - 1 \right]$$

$$= \frac{e^{-1} - e^{-3}}{2} \approx 0.159.$$

b) Find $P(X = 2Y)$.

$$P(X = 2Y) = \sum_{k=1}^{\infty} p_X(2k) \cdot p_Y(k) = \sum_{k=1}^{\infty} \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right)^{2k-1} \cdot \frac{3^k e^{-3}}{k!}$$

$$= \frac{e^{-3}}{2} \cdot \sum_{k=1}^{\infty} \left(\frac{4}{3}\right)^k \cdot \frac{1}{k!} = \frac{e^{-3}}{2} \cdot \left[e^{4/3} - 1 \right] \approx 0.069544.$$

For fun:

c) $$P(X > Y) = \sum_{y=0}^{\infty} \sum_{x=y+1}^{\infty} \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right)^{x-1} \cdot \frac{3^y e^{-3}}{y!} = \sum_{y=0}^{\infty} \left(\frac{2}{3}\right)^y \cdot \frac{3^y e^{-3}}{y!}$$

$$= e^{-3} \cdot \sum_{y=0}^{\infty} \frac{2^y}{y!} = e^{-1}.$$