

4.2 Covariance and Correlation CoefficientCovariance of X and Y

$$\sigma_{XY} = \text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = E(XY) - \mu_X \mu_Y$$

- (a) $\text{Cov}(X, X) = \text{Var}(X)$;
- (b) $\text{Cov}(X, Y) = \text{Cov}(Y, X)$;
- (c) $\text{Cov}(aX + b, Y) = a \text{Cov}(X, Y)$;
- (d) $\text{Cov}(X + Y, W) = \text{Cov}(X, W) + \text{Cov}(Y, W)$.

$$\begin{aligned} \text{Cov}(aX + bY, cX + dY) \\ = ac \text{Var}(X) + (ad + bc) \text{Cov}(X, Y) + bd \text{Var}(Y). \end{aligned}$$

$$\begin{aligned} \text{Var}(aX + bY) &= \text{Cov}(aX + bY, aX + bY) \\ &= a^2 \text{Var}(X) + 2ab \text{Cov}(X, Y) + b^2 \text{Var}(Y). \end{aligned}$$

0. Find in terms of σ_X^2 , σ_Y^2 , and σ_{XY} :

- a)
- $\text{Cov}(2X + 3Y, X - 2Y)$
- ,

$$\text{Cov}(2X + 3Y, X - 2Y) = 2 \text{Var}(X) - \text{Cov}(X, Y) - 6 \text{Var}(Y).$$

- b)
- $\text{Var}(2X + 3Y)$
- ,

$$\begin{aligned} \text{Var}(2X + 3Y) &= \text{Cov}(2X + 3Y, 2X + 3Y) \\ &= 4 \text{Var}(X) + 12 \text{Cov}(X, Y) + 9 \text{Var}(Y). \end{aligned}$$

- c)
- $\text{Var}(X - 2Y)$
- .

$$\begin{aligned} \text{Var}(X - 2Y) &= \text{Cov}(X - 2Y, X - 2Y) \\ &= \text{Var}(X) - 4 \text{Cov}(X, Y) + 4 \text{Var}(Y). \end{aligned}$$

Correlation coefficient of X and Y

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)} \cdot \sqrt{\text{Var}(Y)}} = E \left[\left(\frac{X - \mu_X}{\sigma_X} \right) \left(\frac{Y - \mu_Y}{\sigma_Y} \right) \right]$$

- (a) $-1 \leq \rho_{XY} \leq 1$;
- (b) ρ_{XY} is either +1 or -1 if and only if X and Y are linear functions of one another.

If random variables X and Y are independent, then

$$E(g(X) \cdot h(Y)) = E(g(X)) \cdot E(h(Y)).$$

$$\Rightarrow \text{Cov}(X, Y) = \sigma_{XY} = 0, \quad \text{Corr}(X, Y) = \rho_{XY} = 0.$$

1. Consider the following joint probability distribution $p(x, y)$ of two random variables X and Y:

	y			
x	0	1	2	$p_X(x)$
1	0.15	0.10	0	0.25
2	0.25	0.30	0.20	0.75
$p_Y(y)$	0.40	0.40	0.20	1.00

Recall:

$$E(X) = 1.75,$$

$$E(Y) = 0.8,$$

$$E(XY) = 1.5.$$

Find $\text{Cov}(X, Y) = \sigma_{XY}$ and $\text{Corr}(X, Y) = \rho_{XY}$.

$$\text{Cov}(X, Y) = \sigma_{XY} = 1.5 - 1.75 \cdot 0.8 = \mathbf{0.10}.$$

$$E(X^2) = 1 \times 0.25 + 4 \times 0.75 = 3.25.$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 3.25 - 1.75^2 = 0.1875.$$

$$E(Y^2) = 0 \times 0.40 + 1 \times 0.40 + 4 \times 0.20 = 1.2.$$

$$\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = 1.2 - 0.8^2 = 0.56.$$

$$\text{Corr}(X, Y) = \rho_{XY} = \frac{0.10}{\sqrt{0.1875} \cdot \sqrt{0.56}} \approx \mathbf{0.3086}.$$

2. Let the joint probability density function for (X, Y) be

$$f(x, y) = \begin{cases} 60x^2y & 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $\text{Cov}(X, Y) = \sigma_{XY}$ and $\text{Corr}(X, Y) = \rho_{XY}$.

Recall: $f_X(x) = 30x^2(1-x)^2, \quad 0 < x < 1, \quad E(X) = \frac{1}{2},$

$f_Y(y) = 20y(1-y)^3, \quad 0 < y < 1, \quad E(Y) = \frac{1}{3}, \quad E(XY) = \frac{1}{7}.$

$$\text{Cov}(X, Y) = \frac{1}{7} - \frac{1}{2} \cdot \frac{1}{3} = -\frac{1}{42}. \quad \text{Var}(X) = \frac{9}{252}. \quad \text{Var}(Y) = \frac{8}{252}.$$

$$\rho_{XY} = \frac{-1/42}{\sqrt{9/252} \cdot \sqrt{8/252}} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \approx -\mathbf{0.7071}.$$

3. Let the joint probability density function for (X, Y) be

$$f(x, y) = \begin{cases} x + y & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $\text{Cov}(X, Y) = \sigma_{XY}$ and $\text{Corr}(X, Y) = \rho_{XY}$.

Recall: $f_X(x) = x + \frac{1}{2}, \quad 0 < x < 1. \quad f_Y(y) = y + \frac{1}{2}, \quad 0 < y < 1.$

$$\mu_x = \int_0^1 x \left(x + \frac{1}{2} \right) dx = \left[\frac{1}{3}x^3 + \frac{1}{4}x^2 \right]_0^1 = \frac{7}{12};$$

$$\mu_y = \int_0^1 y \left(y + \frac{1}{2} \right) dy = \frac{7}{12};$$

$$E(X^2) = \int_0^1 x^2 \left(x + \frac{1}{2} \right) dx = \left[\frac{1}{4}x^4 + \frac{1}{6}x^3 \right]_0^1 = \frac{5}{12},$$

$$\sigma_x^2 = E(X^2) - \mu_x^2 = \frac{5}{12} - \left(\frac{7}{12} \right)^2 = \frac{11}{144}.$$

Similarly, $\sigma_y^2 = \frac{11}{144}.$

$$\begin{aligned}
 E(XY) &= \int_0^1 \int_0^1 xy(x+y) dx dy = \int_0^1 \int_0^1 (x^2 y + x y^2) dx dy \\
 &= \int_0^1 \left(\frac{y}{3} + \frac{y^2}{2} \right) dy = \frac{1}{3}.
 \end{aligned}$$

$$\text{Cov}(X, Y) = \frac{1}{3} - \frac{7}{12} \cdot \frac{7}{12} = -\frac{\mathbf{1}}{\mathbf{144}}. \quad \rho_{XY} = \frac{-1/144}{\sqrt{11/144} \cdot \sqrt{11/144}} = -\frac{\mathbf{1}}{\mathbf{11}}.$$

4. Let the joint probability density function for (X, Y) be

$$f(x, y) = \begin{cases} 12x(1-x)e^{-2y} & 0 \leq x \leq 1, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $\text{Cov}(X, Y) = \sigma_{XY}$ and $\text{Corr}(X, Y) = \rho_{XY}$.

Recall: $f_X(x) = 6x(1-x), 0 < x < 1.$ $f_Y(y) = 2e^{-2y}, y > 0.$

Since X and Y are independent,

$$\text{Cov}(X, Y) = \sigma_{XY} = \mathbf{0}, \quad \text{Corr}(X, Y) = \rho_{XY} = \mathbf{0}.$$