STAT 400
UIUC

Answers for 5.3, 5.4, 5.5

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1. Models of the pricing of stock options often make the assumption of a normal distribution. An investor believes that the price of an *Burger Queen* stock option is a normally distributed random variable with mean \$18 and standard deviation \$3. He also believes that the price of an *Dairy King* stock option is a normally distributed random variable with mean \$14 and standard deviation \$2. Assume the stock options of these two companies are independent. The investor buys 8 shares of *Burger Queen* stock option and 9 shares of *Dairy King* stock option. What is the probability that the value of this portfolio will exceed \$300?

BQ has Normal distribution, $\mu_{BQ} = \$18$, $\sigma_{BQ} = \$3$.

DK has Normal distribution, $\mu_{DK} = \$14$, $\sigma_{DK} = \$2$.

Value of the portfolio VP = 8 × BQ + 9 × DK.

Then VP has Normal distribution.

$\mu_{VP} = 8 \times \mu_{BQ} + 9 \times \mu_{DK} = 8 \times 18 + 9 \times 14 = \$270$.

$\sigma_{VP}^2 = 8^2 \times \sigma_{BQ}^2 + 9^2 \times \sigma_{DK}^2 = 64 \times 9 + 81 \times 4 = 900$. $\sigma_{VP} = \$30$.

$P(\ VP > 300\ ) = P\left(Z > \frac{300 - 270}{30}\right) = P(\ Z > 1.00\ ) = 1 - 0.8413 = \mathbf{0.1587}$.

2. A machine fastens plastic screw-on caps onto containers of motor oil. If the machine applies more torque than the cap can withstand, the cap will break. Both the torque applied and the strength of the caps vary. The capping machine torque has the normal distribution with mean 7.9 inch-pounds and standard deviation 0.9 inch-pounds. The cap strength (the torque that would break the cap) has the normal distribution with mean 10 inch-pounds and standard deviation 1.2 inch-pounds. The cap strength and the torque applied by the machine are independent. What is the probability that a cap will break while being fastened by the capping machine? That is, find the probability $\mathrm{P}(\text{Strength} < \text{Torque})$.

$$\mathrm{P}(\text{Strength} < \text{Torque}) = \mathrm{P}(\text{Strength} - \text{Torque} < 0).$$

$$\mathrm{E}(\text{Strength} - \text{Torque}) = 10 - 7.9 = 2.1.$$

$$\mathrm{Var}(\text{Strength} - \text{Torque}) = (1)^2\, 1.2^2 + (-1)^2\, 0.9^2 = 2.25.$$

$$\mathrm{SD}(\text{Strength} - \text{Torque}) = 1.5.$$

$(\text{Strength} - \text{Torque})$ is normally distributed.

$$\mathrm{P}(\text{Strength} - \text{Torque} < 0) = \mathrm{P}\left(Z < \frac{0 - 2.1}{1.5}\right) = \mathrm{P}(Z < -1.40)$$

$$= \Phi(-1.40) = \mathbf{0.0808}.$$

3. In Neverland, the weights of adult men are normally distributed with mean of 170 pounds and standard deviation of 10 pounds, and the weights of adult women are normally distributed with mean of 125 pounds and standard deviation of 8 pounds. Six women and four men got on an elevator. Assume that all their weights are independent. What is the probability that their total weight exceeds 1500 pounds?

Total = $W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + M_1 + M_2 + M_3 + M_4$.

$$\begin{aligned} E(\text{Total}) &= E(W_1) + E(W_2) + E(W_3) + E(W_4) + E(W_5) + E(W_6) \\ &\quad + E(M_1) + E(M_2) + E(M_3) + E(M_4) \\ &= 125 + 125 + 125 + 125 + 125 + 125 + 170 + 170 + 170 + 170 = 1430. \end{aligned}$$

$$\begin{aligned} \text{Var}(\text{Total}) &= \text{Var}(W_1) + \text{Var}(W_2) + \text{Var}(W_3) + \text{Var}(W_4) + \text{Var}(W_5) + \text{Var}(W_6) \\ &\quad + \text{Var}(M_1) + \text{Var}(M_2) + \text{Var}(M_3) + \text{Var}(M_4) \\ &= 64 + 64 + 64 + 64 + 64 + 64 + 100 + 100 + 100 + 100 = 784. \end{aligned}$$

SD(Total) = $\sqrt{784} = 28$. Total has Normal distribution.

$$P(\text{ Total} > 1500 \text{ }) = P\left(Z > \frac{1500 - 1430}{28} \right) = P(\text{ } Z > 2.50 \text{ }) = 1 - 0.9938 = \mathbf{0.0062}.$$

Note: It is tempting to set Total = 6 W + 4 M, but that would imply that the six women who got on the elevator all have the same weight, and so do the four men, which is most likely not the case here.

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4. Let X and Y be two independent Poisson random variables with mean λ_1 and λ_2, respectively. Let W = X + Y.

a) What is the probability distribution of W?

$$\mathrm{P}(\mathrm{W}=n) = \sum_{k=0}^{n} \mathrm{P}(\mathrm{X}=k)\cdot \mathrm{P}(\mathrm{Y}=n-k) = \sum_{k=0}^{n} \frac{\lambda_1^k \cdot e^{-\lambda_1}}{k!}\cdot\frac{\lambda_2^{n-k}\cdot e^{-\lambda_2}}{(n-k)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}\cdot\sum_{k=0}^{n}\frac{n!}{k!\cdot(n-k)!}\cdot\lambda_1^k\cdot\lambda_2^{n-k} = \frac{(\lambda_1+\lambda_2)^n\cdot e^{-(\lambda_1+\lambda_2)}}{n!}.$$

Therefore, W is a Poisson random variable with mean $\lambda_1+\lambda_2$.

OR

$$\mathrm{M}_{\mathrm{W}}(t) = \mathrm{M}_{\mathrm{X}}(t)\cdot\mathrm{M}_{\mathrm{Y}}(t) = e^{\lambda_1(e^t-1)}\cdot e^{\lambda_2(e^t-1)} = e^{(\lambda_1+\lambda_2)(e^t-1)}.$$

Therefore, W is a Poisson random variable with mean $\lambda_1+\lambda_2$.

b)* What is the conditional distribution of X given, W = n?

$$\mathrm{P}(\mathrm{X}=k\mid\mathrm{W}=n) = \frac{\mathrm{P}(\mathrm{X}=k\cap\mathrm{W}=n)}{\mathrm{P}(\mathrm{W}=n)} = \frac{\mathrm{P}(\mathrm{X}=k\cap\mathrm{Y}=n-k)}{\mathrm{P}(\mathrm{W}=n)}$$

$$= \frac{\dfrac{\lambda_1^k\cdot e^{-\lambda_1}}{k!}\cdot\dfrac{\lambda_2^{n-k}\cdot e^{-\lambda_2}}{(n-k)!}}{\dfrac{(\lambda_1+\lambda_2)^n\cdot e^{-(\lambda_1+\lambda_2)}}{n!}}$$

$$= \frac{n!}{k!\cdot(n-k)!}\cdot\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\cdot\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}.$$

$\Rightarrow$ X | W = n has a Binomial distribution, $p = \dfrac{\lambda_1}{\lambda_1+\lambda_2}$.

5. Let X and Y be independent random variables, each geometrically distributed with the probability of "success" p, $0 < p < 1$. That is,

$$p_{\mathrm{X}}(k) = p_{\mathrm{Y}}(k) = p \cdot (1-p)^{k-1}, \qquad k = 1, 2, 3, \ldots,$$

a) Find $\mathrm{P}(\mathrm{X} + \mathrm{Y} = n)$, $n = 2, 3, 4, \ldots$.

$$\mathrm{P}(\mathrm{X} + \mathrm{Y} = n) = \sum_{k=1}^{n-1} \mathrm{P}(\mathrm{X} = k) \cdot \mathrm{P}(\mathrm{Y} = n-k)$$

$$= \sum_{k=1}^{n-1} p \cdot (1-p)^{k-1} \cdot p \cdot (1-p)^{n-k-1} = \sum_{k=1}^{n-1} p^2 \cdot (1-p)^{n-2}$$

$$= (n-1) \cdot p^2 \cdot (1-p)^{n-2}, \qquad n = 2, 3, 4, \ldots.$$

If X and Y both have Geometric(p) distribution and are independent, then X + Y has Negative Binomial distribution with $r = 2$.

OR

$$\mathrm{M}_{\mathrm{X+Y}}(t) = \mathrm{M}_{\mathrm{X}}(t) \cdot \mathrm{M}_{\mathrm{Y}}(t) = \left[\frac{p\,e^{-t}}{1 - (1-p)\,e^{-t}} \right]^2.$$

b) Find $\mathrm{P}(\mathrm{X} = k \mid \mathrm{X} + \mathrm{Y} = n)$, $k = 1, 2, 3, \ldots, n-1$, $n = 2, 3, 4, \ldots$.

$$\mathrm{P}(\mathrm{X} = k \mid \mathrm{X} + \mathrm{Y} = n) = \frac{\mathrm{P}(\mathrm{X} = k \cap \mathrm{X} + \mathrm{Y} = n)}{\mathrm{P}(\mathrm{X} + \mathrm{Y} = n)} = \frac{\mathrm{P}(\mathrm{X} = k \cap \mathrm{Y} = n-k)}{\mathrm{P}(\mathrm{X} + \mathrm{Y} = n)}$$

$$= \frac{p \cdot (1-p)^{k-1} \cdot p \cdot (1-p)^{n-k-1}}{(n-1) \cdot p^2 \cdot (1-p)^{n-2}} = \frac{1}{n-1}, \qquad k = 1, 2, 3, \ldots, n-1.$$

$\Rightarrow$ $\mathrm{X} \mid \mathrm{X} + \mathrm{Y} = n$ has a Uniform distribution on integers $1, 2, 3, \ldots, n-1$.

c) Find $P(X > Y)$. [Hint: First, find $P(X = Y)$.]

$$P(X = Y) = \sum_{k=1}^{\infty} p_X(k) \cdot p_Y(k) = \sum_{k=1}^{\infty} p \cdot (1-p)^{k-1} \cdot p \cdot (1-p)^{k-1}$$

$$= p^2 \cdot \sum_{k=1}^{\infty} \left[(1-p)^2 \right]^{k-1} = p^2 \cdot \sum_{n=0}^{\infty} \left[(1-p)^2 \right]^{n}$$

$$= \frac{p^2}{1-(1-p)^2} = \frac{p}{2-p}.$$

$$P(X > Y) + P(X = Y) + P(X < Y) = 1.$$

Since $P(X > Y) = P(X < Y)$,

$$P(X > Y) = \frac{1}{2} \cdot (1 - P(X = Y)) = \frac{1}{2} \cdot \left(1 - \frac{p}{2-p} \right) = \frac{1-p}{2-p}.$$

OR

$$P(X > Y) = \sum_{y=1}^{\infty} \sum_{x=y+1}^{\infty} p \cdot (1-p)^{x-1} \cdot p \cdot (1-p)^{y-1}$$

$$= \sum_{y=1}^{\infty} p^2 \cdot (1-p)^{y-1} \cdot \sum_{x=y+1}^{\infty} (1-p)^{x-1}$$

$$= \sum_{y=1}^{\infty} p^2 \cdot (1-p)^{y-1} \cdot \frac{(1-p)^y}{1-(1-p)} = \sum_{y=1}^{\infty} p \cdot (1-p)^{2y-1}$$

$$= p \cdot (1-p) \cdot \sum_{n=0}^{\infty} \left[(1-p)^2 \right]^{n} = \frac{p \cdot (1-p)}{1-(1-p)^2} = \frac{1-p}{2-p}.$$

d) Consider the discrete random variable $Q = \frac{X}{Y}$.

Find $E(X)$, $E(\frac{1}{Y})$, $E(Q)$.

[Hint: $\ln(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}$ for $-1 < z < 1$.]

$$E(X) = \frac{1}{p}, \quad \text{since X has a Geometric}(p) \text{ distribution.}$$

$$E(\frac{1}{Y}) = \sum_{k=1}^{\infty} \frac{1}{k} \cdot p \cdot (1-p)^{k-1} = \frac{p}{1-p} \cdot \sum_{k=1}^{\infty} \frac{(1-p)^k}{k}$$

$$= -\ln(1-(1-p)) \cdot \frac{p}{1-p} = -\ln(p) \cdot \frac{p}{1-p}.$$

Since X and Y are independent,

$$E(Q) = E(X) \times E(\frac{1}{Y}) = \frac{-\ln(p)}{1-p}.$$

e) For any positive, irreducible fraction $\frac{a}{b}$, find $P(Q = \frac{a}{b})$.

$$P(Q = \frac{a}{b}) = \sum_{k=1}^{\infty} p_X(ka) \cdot p_Y(kb)$$

$$= \sum_{k=1}^{\infty} p \cdot (1-p)^{ka-1} \cdot p \cdot (1-p)^{kb-1}$$

$$= \left(\frac{p}{1-p}\right)^2 \cdot \sum_{k=1}^{\infty} \left[(1-p)^{a+b}\right]^k$$

$$= \frac{p^2}{(1-p)^2} \cdot \frac{(1-p)^{a+b}}{1-(1-p)^{a+b}}.$$