STAT 400
UIUC

Answers for 5.6

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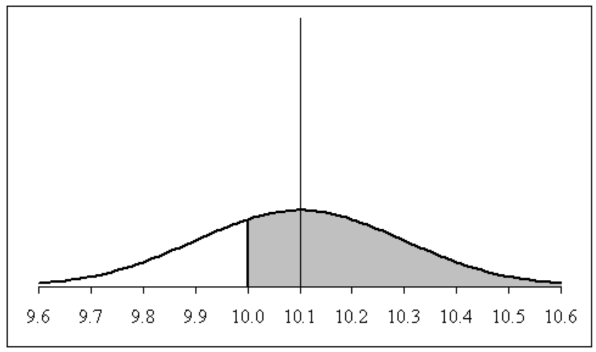
½. The true weight of "10-pound" sacks of potatoes processed at a certain packaging house follows a normal distribution with mean of 10.1 pounds and standard deviation of 0.2 pounds.

a) What is the probability that a sack weighs at least 10 pounds?

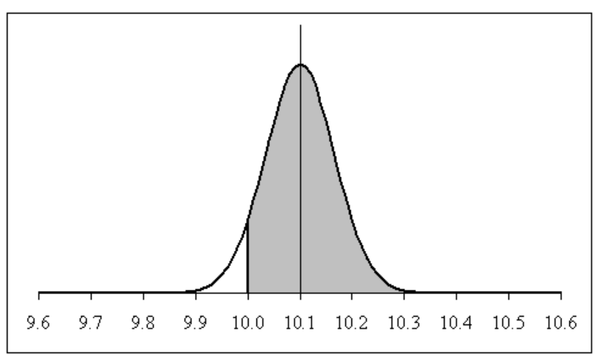
$$P(X \ge 10) = P\left(Z \ge \frac{10 - 10.1}{0.2}\right)$$
$$= P(Z \ge -0.50)$$
$$= 1 - \Phi(-0.50)$$
$$= 1 - 0.3085$$
$$= \mathbf{0.6915}.$$



b) A random sample of 9 sacks is selected. What is the probability that the average weight of these 9 is at least 10 pounds?

Case 2. $$\frac{\overline{X} - \mu}{\sigma / \sqrt{n}} = Z.$$

$$P(\overline{X} \ge 10) = P\left(Z \ge \frac{10 - 10.1}{0.2 / \sqrt{9}}\right)$$
$$= P(Z \ge -1.50)$$
$$= 1 - \Phi(-1.50)$$
$$= 1 - 0.0668$$
$$= \mathbf{0.9332}.$$



1. A student commission wants to know the mean amount of money spent by college students for textbooks in one semester. Suppose the population mean is \$450 and the population standard deviation is \$40. A random sample of 625 students is taken.

a) What is the probability that the sample mean will be less than \$452?

$\mu = 450$, $\sigma = 40$, $n = 625$. Need $P(\overline{X} < 452) = ?$

$n = 625$ – large. Central Limit Theorem: $\dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} \approx Z$.

$$P(\overline{X} < 452) \approx P\left(Z < \frac{452 - 450}{40 / \sqrt{625}}\right) = P(Z < 1.25) = \mathbf{0.8944}.$$

b) What is the probability that the sample mean will be within \$2 of \$450? That is, what is the probability that the sample mean will be between \$448 and \$452?

$$P(448 < \overline{X} < 452) \approx P\left(\frac{448 - 450}{40 / \sqrt{625}} < Z < \frac{452 - 450}{40 / \sqrt{625}}\right) = P(-1.25 < Z < 1.25) = \mathbf{0.7888}.$$

c) What is the probability that the sample mean will be within \$10 of \$450? That is, what is the probability that the sample mean will be between \$440 and \$460?

$$P(440 < \overline{X} < 460) \approx P\left(\frac{440 - 450}{40 / \sqrt{625}} < Z < \frac{460 - 450}{40 / \sqrt{625}}\right) = P(-6.25 < Z < 6.25) \approx \mathbf{1.00}.$$

2. The amount of sulfur in the daily emissions from a power plant has a normal distribution with mean of 134 pounds and a standard deviation of 22 pounds. For a random sample of 5 days, find the probability that the total amount of sulfur emissions will exceed 700 pounds.

Normal distribution, $\mu = 134$, $\sigma = 22$, $n = 5$.

Need $P(X_1 + X_2 + X_3 + X_4 + X_5 > 700) = P(\text{Total} > 700) = ?$

Total has $N\left(n\mu, n\sigma^2\right)$ distribution. $\Rightarrow$ $\dfrac{\text{Total} - n\mu}{\sqrt{n}\,\sigma} = Z$.

$$P(\text{Total} > 700) = P\left(Z > \frac{700 - 5 \cdot 134}{\sqrt{5} \cdot 22} \right) = P(Z > 0.61) = 1 - 0.7291 = \mathbf{0.2709}.$$

OR

Need $P(X_1 + X_2 + X_3 + X_4 + X_5 > 700) = P(\overline{X} > 140) = ?$

$\overline{X}$ has $N\left(\mu, \dfrac{\sigma^2}{n}\right)$ distribution. $\Rightarrow$ $\dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} = Z$.

$$P(\overline{X} > 140) = P\left(Z > \frac{140 - 134}{22 / \sqrt{5}} \right) = P(Z > 0.61) = 1 - 0.7291 = \mathbf{0.2709}.$$

3. An economist wishes to estimate the average family income in a certain population. The population standard deviation is known to be $4,500, and the economist uses a random sample of 225 families. What is the probability that the sample mean will fall within $600 of the population mean?

$\mu = ?$, $\sigma = 4{,}500$, $n = 225$. Need $P(\mu - 600 < \overline{X} < \mu + 600) = ?$

$n = 225$ – large. Central Limit Theorem: $\dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} = Z$.

$$P(\mu - 600 < \overline{X} < \mu + 600) = P\left(\frac{(\mu - 600) - \mu}{4{,}500 / \sqrt{225}} < Z < \frac{(\mu + 600) - \mu}{4{,}500 / \sqrt{225}} \right)$$

$$= P(-2.00 < Z < 2.00) = 0.9772 - 0.0228 = \mathbf{0.9544}.$$

4. Forty-eight measurements are recorded to several decimal places. Each of these 48 numbers is rounded off to the nearest integer. The sum of the original 48 numbers is approximated by the sum of these integers. If we assume that the errors made by rounding off are i.i.d. and have uniform distribution over the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$, compute approximately the probability that the sum of the integers is within 2 units of the true sum.

$X_1, X_2, \ldots, X_{48}$ are i.i.d. Uniform$\left(-\frac{1}{2}, \frac{1}{2}\right)$. $\mu_X = 0$, $\sigma_X^2 = \dfrac{1}{12}$.

$n = 48$ – large. Central Limit Theorem: $\dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} = Z$.

$$\text{Need } P(|X_1 + X_2 + \ldots + X_{48}| < 2) = P\left(|\overline{X}| < \frac{2}{48}\right) \approx P\left(|Z| < \frac{2/48}{\sqrt{(1/12)/48}}\right)$$

$$= P(|Z| < 1.00) = 0.8413 - 0.1587 = \mathbf{0.6826}.$$