Answers for 6.4, Part 2

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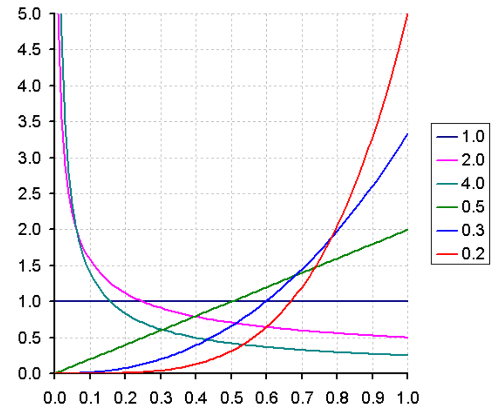
4. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the distribution with probability density function

$$f(x;\theta) = \begin{cases} \frac{1}{\theta} \cdot x^{\frac{1-\theta}{\theta}} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

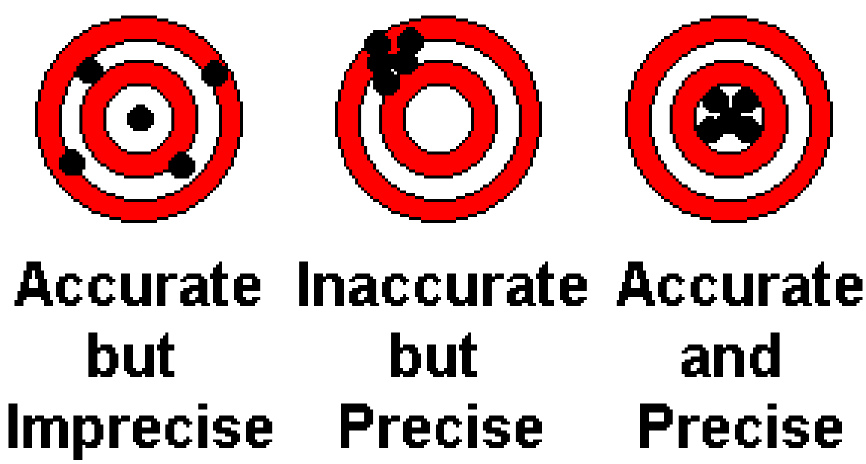
$0 < \theta < \infty$.



Recall: Maximum likelihood estimator of θ is $\hat{\theta} = -\frac{1}{n} \cdot \sum_{i=1}^{n} \ln X_i$.

Method of moments estimator of θ is $\tilde{\theta} = \frac{1-\overline{X}}{\overline{X}} = \frac{1}{\overline{X}} - 1$. $\quad E(X) = \frac{1}{1+\theta}$.

Def An estimator $\hat{\theta}$ is said to be **unbiased for θ** if $E(\hat{\theta}) = \theta$ for all θ.



unbiased,	biased,	unbiased,
large variance	small variance	small variance

d) Is $\hat{\theta}$ unbiased for θ? That is, does $E(\hat{\theta})$ equal θ?

$$E(\ln X_1) = \int_{-\infty}^{\infty} \ln x \cdot f_X(x;\theta)\,dx = \int_0^1 \left(\ln x \cdot \frac{1}{\theta} \cdot x^{1-\theta \,/\, \theta} \right) dx.$$

Integration by parts:

$$\int_a^b u\,dv = u\,v \Big|_a^b - \int_a^b v\,du$$

Choice of u:

L ogarithmic
A lgebraic
T rigonometric
E xponential

$$u = \ln x, \qquad dv = \frac{1}{\theta} \cdot x^{1-\theta \,/\, \theta}\,dx = \frac{1}{\theta} \cdot x^{\frac{1}{\theta}-1}\,dx,$$

$$du = \frac{1}{x}\,dx, \qquad v = x^{1/\theta}.$$

$$E(\ln X_1) = \int_0^1 \left(\ln x \cdot \frac{1}{\theta} \cdot x^{1-\theta \,/\, \theta} \right) dx = \left(\ln x \cdot x^{1/\theta} \right) \Bigg|_0^1 - \int_0^1 \left(\frac{1}{x} \cdot x^{1/\theta} \right) dx$$

$$= -\int_0^1 \left(\frac{1}{x} \cdot x^{1/\theta} \right) dx = -\int_0^1 x^{\frac{1}{\theta}-1}\,dx = -\left(\frac{1}{1/\theta} \cdot x^{1/\theta} \right) \Bigg|_0^1 = -\theta.$$

Therefore,

$$E(\hat{\theta}) = -\frac{1}{n} \cdot \sum_{i=1}^{n} E(\ln X_i) = -\frac{1}{n} \cdot \sum_{i=1}^{n} (-\theta) = \theta,$$

that is, $\hat{\theta}$ is an unbiased estimator for θ.

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Jensen's Inequality:

If g is convex on an open interval I and X is a random variable whose support is contained in I and has finite expectation, then

$$E[g(X)] \geq g[E(X)].$$

If g is strictly convex then the inequality is strict, unless X is a constant random variable.

$\Rightarrow \quad E(X^2) \geq [E(X)]^2 \qquad \Leftrightarrow \qquad Var(X) \geq 0$

$\Rightarrow \quad E(e^{tX}) \geq e^{tE(X)} \qquad \Rightarrow \qquad M_X(t) \geq e^{t\mu}$

$\Rightarrow \quad E\left(\frac{1}{X}\right) \geq \frac{1}{E(X)}$ for a positive random variable X

$\Rightarrow \quad E[X^3] \geq [E(X)]^3$ for a non-negative random variable X

$\Rightarrow \quad E[\ln X] \leq \ln E(X)$ for a positive random variable X

$\Rightarrow \quad E\left(\sqrt{X}\right) \leq \sqrt{E(X)}$ for a non-negative random variable X

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e) Is $\tilde{\theta}$ unbiased for θ? That is, does $E(\tilde{\theta})$ equal θ?

Since $g(x) = \frac{1-x}{x} = \frac{1}{x} - 1$, $0 < x < 1$, is strictly convex, and $\overline{X}$ is not a constant random variable, by Jensen's Inequality,

$$E(\tilde{\theta}) = E(g(\overline{X})) > g(E(\overline{X})) = \theta.$$

$\tilde{\theta}$ is NOT an unbiased estimator for θ.

6. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a population with mean μ and variance σ^2. Show that the sample mean $\overline{X}$ and the sample variance S^2 are unbiased for μ and σ^2, respectively.

$$\overline{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

$$E(X_1 + X_2 + \ldots + X_n) = n \cdot \mu \quad \Rightarrow \quad E(\overline{X}) = \mu \quad \checkmark$$

$$E(X^2) = \mathrm{Var}(X) + [E(X)]^2 = \mu^2 + \sigma^2.$$

$$\mathrm{Var}(X_1 + X_2 + \ldots + X_n) = n \cdot \sigma^2 \quad \Rightarrow \quad \mathrm{Var}(\overline{X}) = \sigma^2/n$$

$$E\left(\left(\overline{X}\right)^2\right) = \mathrm{Var}(\overline{X}) + [E(\overline{X})]^2 = \mu^2 + \frac{\sigma^2}{n}.$$

$$S^2 = \frac{1}{n-1}\sum\left(X_i - \overline{X}\right)^2 = \frac{1}{n-1}\left[\sum X_i^2 - n \cdot \left(\overline{X}\right)^2\right]$$

$$E(S^2) = \frac{1}{n-1}\left[\sum E\left(X_i^2\right) - n \cdot E\left(\left(\overline{X}\right)^2\right)\right]$$

$$= \frac{1}{n-1}\left[n \cdot \left(\mu^2 + \sigma^2\right) - n \cdot \left(\mu^2 + \frac{\sigma^2}{n}\right)\right] = \sigma^2 \quad \checkmark$$

For an estimator $\hat{\theta}$ of θ, define the **Mean Squared Error** of $\hat{\theta}$ by

$$\mathrm{MSE}(\hat{\theta}) = E[(\hat{\theta} - \theta)^2].$$

$$E[(\hat{\theta} - \theta)^2] = \left(E(\hat{\theta}) - \theta\right)^2 + \mathrm{Var}(\hat{\theta}) = \left(\mathrm{bias}(\hat{\theta})\right)^2 + \mathrm{Var}(\hat{\theta}).$$

7. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a distribution with probability density function

$$f_X(x) = f_X(x;\theta) = \frac{1+\theta x}{2}, \qquad -1 < x < 1, \qquad -1 < \theta < 1.$$

a) Obtain the method of moments estimator of θ, $\tilde{\theta}$.

$$\mu = E(X) = \int_{-1}^{1} x \cdot \frac{1+\theta x}{2}\, dx = \left(\frac{x^2}{4} + \frac{\theta x^3}{6} \right) \Bigg|_{-1}^{1} = \frac{\theta}{3}.$$

$$\overline{X} = \frac{\tilde{\theta}}{3} \qquad \Rightarrow \qquad \tilde{\theta} = 3\,\overline{X}.$$

b) Is $\tilde{\theta}$ an unbiased estimator for θ? *Justify your answer.*

$$E(\tilde{\theta}) = E(3\,\overline{X}) = 3\,E(\overline{X}) = 3\,\mu = 3\,\frac{\theta}{3} = \theta.$$

$\Rightarrow$ $\tilde{\theta}$ an unbiased estimator for θ.

c) Find $\operatorname{Var}(\tilde{\theta})$.

$$E(X^2) = \int_{-1}^{1} x^2 \cdot \frac{1+\theta x}{2}\, dx = \left(\frac{x^3}{6} + \frac{\theta x^4}{8} \right) \Bigg|_{-1}^{1} = \frac{1}{3}.$$

$$\sigma^2 = \operatorname{Var}(X) = \frac{1}{3} - \left(\frac{\theta}{3}\right)^2 = \frac{3-\theta^2}{9}.$$

$$\operatorname{Var}(\tilde{\theta}) = 9\,\operatorname{Var}(\overline{X}) = 9 \cdot \frac{\sigma^2}{n} = \frac{3-\theta^2}{n}. \qquad \Rightarrow \qquad \operatorname{MSE}(\tilde{\theta}) = \frac{3-\theta^2}{n}.$$

8. Let X_1, X_2 be a random sample of size $n = 2$ from a distribution with probability density function

$$f_X(x) = f_X(x;\theta) = \frac{1+\theta x}{2}, \qquad -1 < x < 1, \qquad\qquad -1 < \theta < 1.$$

Find the maximum likelihood estimator $\hat{\theta}$ of θ.

$$L(\theta) = \frac{1+\theta x_1}{2} \cdot \frac{1+\theta x_2}{2} = \frac{1+\theta(x_1+x_2)+\theta^2 x_1 x_2}{4}$$

$L(\theta)$ is a parabola with vertex at $\dfrac{-b}{2a} = \dfrac{-(x_1+x_2)}{2x_1 x_2}$.

Case 1: $\quad a = x_1 x_2 > 0. \qquad$ Parabola has its "antlers" up.

$\Rightarrow \qquad$ The vertex is the minimum.

Subcase 1: $\quad x_1 > 0, \quad x_2 > 0. \qquad$ Vertex $= -\dfrac{x_1+x_2}{2x_1 x_2} < 0.$

Maximum of $L(\theta)$ on $-1 < \theta < 1$ is at $\hat{\theta} = 1.$

Subcase 2: $\quad x_1 < 0, \quad x_2 < 0. \qquad$ Vertex $= -\dfrac{x_1+x_2}{2x_1 x_2} > 0.$

Maximum of $L(\theta)$ on $-1 < \theta < 1$ is at $\hat{\theta} = -1.$

Case 2: $\quad a = x_1 x_2 < 0. \qquad$ Parabola has its "antlers" down.

$\Rightarrow \qquad$ The vertex is the maximum.

Vertex is at $-\dfrac{x_1+x_2}{2x_1 x_2}$.

Subcase 1: $\quad -\dfrac{x_1+x_2}{2x_1x_2} > 1.$ That is, $x_2 > -\dfrac{x_1}{2x_1+1}.$

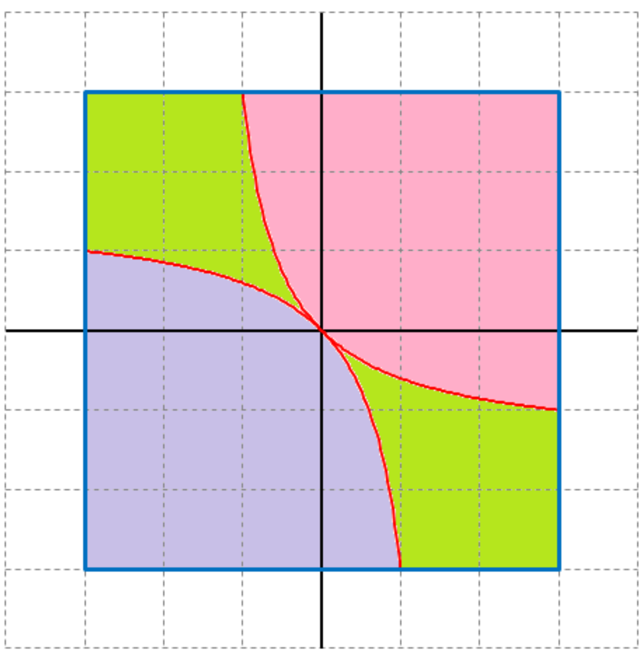
Maximum of $L(\theta)$ on $-1<\theta<1$ is at $\hat{\theta} = 1.$

Subcase 2: $\quad -\dfrac{x_1+x_2}{2x_1x_2} < -1.$ That is, $x_2 < \dfrac{x_1}{2x_1-1}.$

Maximum of $L(\theta)$ on $-1<\theta<1$ is at $\hat{\theta} = -1.$

Subcase 3: $\quad -1 < -\dfrac{x_1+x_2}{2x_1x_2} < 1.$

Maximum of $L(\theta)$ on $-1<\theta<1$ is at $\hat{\theta} = -\dfrac{X_1+X_2}{2X_1X_2}.$



Pink $\quad \hat{\theta} = 1.$

Purple $\quad \hat{\theta} = -1.$

Green $\quad \hat{\theta} = -\dfrac{X_1+X_2}{2X_1X_2}.$

9. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x) = 4\theta x^3 e^{-\theta x^4} \qquad x > 0 \qquad \theta > 0.$$

a) Obtain the maximum likelihood estimator of θ, $\hat{\theta}$.

$$L(\theta) = \prod_{i=1}^{n} \left(4\theta x_i^3 e^{-\theta x_i^4} \right)$$

$$\ln L(\theta) = n \cdot \ln\theta + \sum_{i=1}^{n} \ln\left(4 x_i^3\right) - \theta \cdot \sum_{i=1}^{n} x_i^4$$

$$(\ln L(\theta))' = \frac{n}{\theta} - \sum_{i=1}^{n} x_i^4 = 0 \qquad \Rightarrow \qquad \hat{\theta} = \frac{n}{\sum_{i=1}^{n} X_i^4}.$$

b) Find $E(X^k)$, $k > -4$.

$$E(X^k) = \int_0^\infty x^k \, 4\theta x^3 e^{-\theta x^4} \, dx \qquad u = \theta x^4 \qquad du = 4\theta x^3 dx$$

$$= \int_0^\infty \left(\frac{u}{\theta}\right)^{k/4} e^{-u} \, du = \frac{1}{\theta^{k/4}} \Gamma\left(\frac{k}{4} + 1\right).$$

c) Find the method of moments estimator of θ, $\tilde{\theta}$.

$$E(X) = E(X^1) = \frac{1}{\theta^{1/4}} \Gamma\left(\frac{1}{4} + 1\right) = \frac{1}{\theta^{1/4}} \Gamma(1.25) \approx \frac{0.9064}{\theta^{1/4}}.$$

$$\overline{X} = \frac{\Gamma(1.25)}{\theta^{1/4}}. \qquad \tilde{\theta} = \left(\frac{\Gamma(1.25)}{\overline{X}}\right)^4 \approx \frac{0.675}{\left(\overline{X}\right)^4}.$$