STAT 400
UIUC

Examples for 6.4 (3)

Stepanov
Dalpiaz

1. The Weibull distribution has many applications in reliability engineering, survival analysis, and general insurance. Let $\beta > 0, \delta > 0$. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the distribution with the probability density function

$$f(x;\beta,\delta) = \beta\,\delta\,x^{\delta-1}\,e^{-\beta x^{\delta}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

Suppose δ is known.

a) Obtain the maximum likelihood estimator for β, $\hat{\beta}$.

$$L(\beta) = \prod_{i=1}^{n}\left(\beta\,\delta\,x_i^{\delta-1}\,e^{-\beta x_i^{\delta}}\right).$$

$$\ln L(\beta) = n\cdot\ln\beta + n\cdot\ln\delta + (\delta-1)\cdot\sum_{i=1}^{n}\ln x_i - \beta\cdot\sum_{i=1}^{n}x_i^{\delta}.$$

$$\frac{d}{d\beta}\ln L(\beta) = \frac{n}{\beta} - \sum_{i=1}^{n}x_i^{\delta} = 0. \qquad \Rightarrow \qquad \hat{\beta} = \frac{n}{\sum_{i=1}^{n}X_i^{\delta}}.$$

b) Suppose $\delta = 3$, $n = 5$, and $x_1 = 0.2$, $x_2 = 1.2$, $x_3 = 0.2$, $x_4 = 0.9$, $x_5 = 0.3$. Obtain the maximum likelihood estimate for β, $\hat{\beta}$.

$$\sum_{i=1}^{n}x_i^{3} = 2.5. \qquad \hat{\beta} = \frac{5}{2.5} = \mathbf{2}.$$

Suppose β is known.

c) Obtain an equation for the maximum likelihood estimator for δ, $\hat{\delta}$.

"Hint": $$\frac{d}{d\delta} \ln \mathrm{L}(\delta) = 0.$$

$$\mathrm{L}(\delta) = \prod_{i=1}^{n} \left(\beta\, \delta\, x_i^{\delta-1}\, e^{-\beta x_i^{\delta}} \right).$$

$$\ln \mathrm{L}(\delta) = n \cdot \ln \beta + n \cdot \ln \delta + (\delta - 1) \cdot \sum_{i=1}^{n} \ln x_i - \beta \cdot \sum_{i=1}^{n} x_i^{\delta}.$$

$$\frac{d}{d\delta} \ln \mathrm{L}(\delta) = \frac{n}{\delta} + \sum_{i=1}^{n} \ln x_i - \beta \cdot \sum_{i=1}^{n} x_i^{\delta} \ln x_i = 0.$$

This equation cannot be solved algebraically for δ in closed form.

The solution could be approximated for given $x_1, x_2, \ldots, x_n$ by using iterative numerical procedures.

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d) Find a closed-form expression for $\mathrm{E}(\mathrm{X}^k)$, $k > -\delta$.

"Hint" 1: $u = \beta x^{\delta}$ "Hint" 2: $\Gamma(a) = \int_0^{\infty} u^{a-1} e^{-u}\, du$, $a > 0$.

$$\mathrm{E}(\mathrm{X}^k) = \int_0^{\infty} x^k \beta \delta x^{\delta-1} e^{-\beta x^{\delta}}\, dx \qquad u = \beta x^{\delta} \qquad du = \beta \delta x^{\delta-1}\, dx$$

$$= \int_0^{\infty} \left(\frac{u}{\beta}\right)^{k/\delta} e^{-u}\, du = \frac{1}{\beta^{k/\delta}} \Gamma\left(\frac{k}{\delta} + 1\right).$$

Suppose δ is known.

e) Obtain a method of moments estimator for β, $\tilde{\beta}$.

$$\mathrm{E}(\mathrm{X}) = \frac{1}{\beta^{1/\delta}}\Gamma\left(\frac{1}{\delta}+1\right). \qquad \overline{\mathrm{X}} = \frac{1}{\tilde{\beta}^{1/\delta}}\Gamma\left(\frac{1}{\delta}+1\right).$$

$$\tilde{\beta} = \left(\frac{\Gamma\left(\frac{1}{\delta}+1\right)}{\overline{\mathrm{X}}}\right)^{\delta}.$$

f) Suppose $\delta = 3$, $n = 5$, and $x_1 = 0.2$, $x_2 = 1.2$, $x_3 = 0.2$, $x_4 = 0.9$, $x_5 = 0.3$. Obtain a method of moments estimate for β, $\tilde{\beta}$.

"Hint":	$\Gamma(x)$	R:	`> gamma(x)`
		Excel:	=GAMMA(x)

$$\bar{x} = \frac{2.8}{5} = 0.56. \qquad \Gamma\left(\frac{4}{3}\right) \approx 0.89298.$$

$$\tilde{\beta} = \left(\frac{0.89298}{0.56}\right)^{3} \approx \mathbf{4.0547}.$$

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More on the Method of Moments:

$\mathrm{E}(\mathrm{X}) = h(\theta)$. Set $\overline{\mathrm{X}} = h(\tilde{\theta})$. Solve for $\tilde{\theta}$.

OR

$\mathrm{E}(\mathrm{X}^k) = h(\theta)$. Set $\overline{\mathrm{X}^k} = h(\tilde{\theta})$, where $\overline{\mathrm{X}^k} = \frac{1}{n}\sum_{i=1}^{n}\mathrm{X}_i^k$.

Solve for $\tilde{\theta}$.

OR

e) $E(X^2) = \frac{1}{\beta^{2/\delta}} \Gamma\left(\frac{2}{\delta}+1\right).$ $\overline{X^2} = \frac{1}{\tilde{\beta}^{2/\delta}} \Gamma\left(\frac{2}{\delta}+1\right).$

$$\tilde{\beta} = \left(\frac{\Gamma\left(\frac{2}{\delta}+1\right)}{\overline{X^2}}\right)^{\delta/2}.$$

f) $\overline{x^2} = \frac{2.42}{5} = 0.484.$ $\Gamma\left(\frac{5}{3}\right) \approx 0.902745.$

$$\tilde{\beta}_2 = \left(\frac{0.902745}{0.484}\right)^{1.5} \approx \mathbf{2.5473}.$$

OR

e) $E(X^3) = \frac{1}{\beta^{3/\delta}} \Gamma\left(\frac{3}{\delta}+1\right).$ $\overline{X^3} = \frac{1}{\tilde{\beta}^{3/\delta}} \Gamma\left(\frac{3}{\delta}+1\right).$

$$\tilde{\beta} = \left(\frac{\Gamma\left(\frac{3}{\delta}+1\right)}{\overline{X^3}}\right)^{\delta/3}.$$

f) $\overline{x^3} = \frac{2.5}{5} = 0.50.$ $\Gamma(2) = 1.$

$$\tilde{\beta}_3 = \left(\frac{1}{0.50}\right)^{1} = \mathbf{2}.$$

OR

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2. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f_X(x) = f_X(x;\theta) = \left(\theta^2 + \theta\right) x^{\theta-1}(1-x), \qquad 0 < x < 1, \qquad \theta > 0.$$

a) Obtain the method of moments estimator of θ, $\tilde{\theta}$.

$$E(X) = \int_0^1 x \cdot \left(\theta^2+\theta\right) x^{\theta-1}(1-x)\,dx = \left(\theta^2+\theta\right)\cdot \int_0^1 \left(x^{\theta} - x^{\theta+1}\right) dx$$

$$= \theta\cdot(\theta+1)\cdot\left(\frac{1}{\theta+1}x^{\theta+1} - \frac{1}{\theta+2}x^{\theta+2}\right)\Bigg|_0^1 = \frac{\theta\cdot(\theta+1)}{(\theta+1)\cdot(\theta+2)} = \frac{\theta}{\theta+2}.$$

OR

Beta distribution, $\alpha = \theta$, $\beta = 2$. $\quad\Rightarrow\quad$ $E(X) = \dfrac{\theta}{\theta+2}$.

$$\frac{\tilde{\theta}}{\tilde{\theta}+2} = \overline{X} \qquad \tilde{\theta} = \overline{X}\cdot\left(\tilde{\theta}+2\right) \qquad \tilde{\theta} - \tilde{\theta}\,\overline{X} = 2\overline{X}$$

$$\Rightarrow \quad \tilde{\theta} = \frac{2\overline{X}}{1-\overline{X}}, \qquad \text{where } \overline{X} = \frac{1}{n}\cdot\sum_{i=1}^{n} X_i.$$

b) Suppose $n = 6$, and $x_1 = 0.3$, $x_2 = 0.5$, $x_3 = 0.6$, $x_4 = 0.65$, $x_5 = 0.75$, $x_6 = 0.8$. Find a method of moments estimate of θ.

$x_1 = 0.3$, $x_2 = 0.5$, $x_3 = 0.6$, $x_4 = 0.65$, $x_5 = 0.75$, $x_6 = 0.8$.

$\overline{x} = 0.6.$ $\qquad\qquad$ $\tilde{\theta} = \dfrac{2\overline{x}}{1-\overline{x}} = \mathbf{3}.$

c) Is $\tilde{\theta}$ an unbiased estimator of θ? *Justify your answer.*

Consider $g(x) = \dfrac{2x}{1-x}$. Then $g(\overline{X}) = \tilde{\theta}$, $g(\dfrac{\theta}{\theta+2}) = \theta$.

Also $g''(x) = \dfrac{4}{(1-x)^3} > 0$ for $0 < x < 1$, i.e., $g(x)$ is strictly convex.

By Jensen's Inequality,

$$E(\tilde{\theta}) = E[g(\overline{X})] > g(E(\overline{X})) = g(\mu_X) = g(\frac{\theta}{\theta+2}) = \theta.$$

Therefore, $\tilde{\theta}$ is NOT an unbiased estimator of θ.

d) Obtain the maximum likelihood estimator of θ, $\hat{\theta}$.

That is, find $\hat{\theta} = \arg\max L(\theta) = \arg\max \ln L(\theta)$,

where $L(\theta) = \prod_{i=1}^{n} f(x_i; \theta)$.

"Hint": ① $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$;

② $\theta > 0$;

③ Since $0 < x < 1$, $\ln x < 0$.

$$L(\theta) = \prod_{i=1}^{n} f(x_i; \theta) = f(x_1; \theta)\, f(x_2; \theta)\, \ldots\, f(x_n; \theta)$$

$$= (\theta^2 + \theta)^n \left(\prod_{i=1}^{n} x_i \right)^{\theta - 1} \prod_{i=1}^{n} (1 - x_i).$$

$$\ln L(\theta) = n \ln(\theta^2 + \theta) + (\theta - 1) \sum_{i=1}^{n} \ln x_i + \sum_{i=1}^{n} \ln\left(1 - x_i\right)$$

$$(\ln L(\theta))' = \frac{2n\theta + n}{\theta^2 + \theta} + \sum_{i=1}^{n} \ln x_i = 0.$$

$$\Rightarrow \qquad \Sigma\,\hat{\theta}^2 + (2n + \Sigma)\,\hat{\theta} + n = 0, \qquad \text{where } \Sigma = \sum_{i=1}^{n} \ln X_i .$$

$$\Rightarrow \qquad \hat{\theta} = \frac{-2n - \Sigma \pm \sqrt{(2n + \Sigma)^2 - 4\Sigma n}}{2\Sigma} = \frac{2n + \Sigma \pm \sqrt{4n^2 + \Sigma^2}}{-2\Sigma}.$$

Since $0 < x < 1$, $\ln x < 0$. $\Rightarrow$ $\Sigma < 0.$

$$\Rightarrow \qquad (2n + \Sigma)^2 = 4n^2 + 4n\Sigma + \Sigma^2 < 4n^2 + \Sigma^2.$$

$$\Rightarrow \qquad \left|2n + \Sigma\right| < \sqrt{4n^2 + \Sigma^2}.$$

$\Rightarrow$ Since $\theta > 0$, $$\hat{\theta} = \frac{2n + \Sigma + \sqrt{4n^2 + \Sigma^2}}{-2\Sigma},$$

where $\Sigma = \sum_{i=1}^{n} \ln X_i$.

e) Suppose $n = 6$, and $x_1 = 0.3$, $x_2 = 0.5$, $x_3 = 0.6$, $x_4 = 0.65$, $x_5 = 0.75$, $x_6 = 0.8$. Find the maximum likelihood estimate of θ.

$\Sigma = \sum_{i=1}^{n} \ln X_i \approx -3.349554.$ $\qquad \hat{\theta} \approx \mathbf{3.151}.$

3. Let $\lambda > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a double exponential distribution. That is,

$$f(x;\lambda) = \frac{\lambda}{2} e^{-\lambda |x|}, \qquad -\infty < x < \infty.$$

a) Find $E(X^k)$ for positive integer k.

$$E(X^k) = \int_{-\infty}^{\infty} x^k \cdot \frac{\lambda}{2} e^{-\lambda |x|} dx = \int_{-\infty}^{0} x^k \cdot \frac{\lambda}{2} e^{\lambda x} dx + \int_{0}^{\infty} x^k \cdot \frac{\lambda}{2} e^{-\lambda x} dx = \ldots$$

k odd $\qquad \ldots = 0.$

k even $\qquad \ldots = 2 \cdot \int_{0}^{\infty} x^k \cdot \frac{\lambda}{2} e^{-\lambda x} dx = \int_{0}^{\infty} \lambda \cdot x^k e^{-\lambda x} dx$

$$= \frac{\Gamma(k+1)}{\lambda^k} \cdot \int_{0}^{\infty} \frac{\lambda^{k+1}}{\Gamma(k+1)} x^{k+1-1} e^{-\lambda x} dx = \frac{\Gamma(k+1)}{\lambda^k} = \frac{k!}{\lambda^k}.$$

b) Obtain the maximum likelihood estimator of λ, $\hat{\lambda}$.

$$L(\lambda) = \frac{\lambda^n}{2^n} \exp\left\{ -\lambda \cdot \sum_{i=1}^{n} |x_i| \right\}. \qquad \ln L(\lambda) = n \ln \lambda - n \ln 2 - \lambda \cdot \sum_{i=1}^{n} |x_i|.$$

$$\frac{d}{d\lambda} \ln L(\lambda) = \frac{n}{\lambda} - \sum_{i=1}^{n} |x_i| = 0. \qquad \Rightarrow \qquad \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} |X_i|}.$$

4. Let $\lambda > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with the probability density function

$$f(x;\lambda) = \lambda^3 x^5 e^{-\lambda x^2}, \qquad x > 0.$$

a) Find $E(X^k)$, $k > -6$. "Hint": Consider $u = \lambda x^2$.

$$E(X^k) = \int_0^\infty x^k \cdot \lambda^3 x^5 e^{-\lambda x^2} dx \qquad u = \lambda x^2 \qquad du = 2\lambda x$$

$$= \frac{\lambda^2}{2} \cdot \int_0^\infty \left(\frac{u}{\lambda}\right)^{2+\frac{k}{2}} e^{-u} du = \frac{1}{2}\lambda^{-k/2} \cdot \int_0^\infty u^{2+k/2} e^{-u} du$$

$$= \frac{1}{2}\lambda^{-k/2}\Gamma\left(3+\frac{k}{2}\right).$$

b) Obtain the method of moments estimator of λ, $\tilde{\lambda}$. Suppose $n = 4$, and $x_1 = 4$, $x_2 = 2$, $x_3 = 4$, $x_4 = 3$. Find the method of moments estimate of λ.

$$E(X) = \frac{1}{2}\lambda^{-1/2}\Gamma\left(3+\frac{1}{2}\right) = \frac{1}{2}\lambda^{-1/2} \cdot \frac{5}{2} \cdot \Gamma\left(2+\frac{1}{2}\right) = \frac{1}{2}\lambda^{-1/2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \Gamma\left(1+\frac{1}{2}\right)$$

$$= \frac{1}{2}\lambda^{-1/2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\lambda^{-1/2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} = \frac{15}{16} \cdot \sqrt{\frac{\pi}{\lambda}}.$$

$$\frac{15}{16} \cdot \sqrt{\frac{\pi}{\lambda}} = \overline{X} \qquad \Rightarrow \qquad \tilde{\lambda} = \frac{225\pi}{256\left(\overline{X}\right)^2}.$$

$x_1 = 4$, $x_2 = 2$, $x_3 = 4$, $x_4 = 3$. $\qquad \bar{x} = \frac{13}{4} = 3.25.$

$$\tilde{\lambda} = \frac{225\pi}{256\left(\overline{X}\right)^2} = \frac{225\pi}{2704} \approx \mathbf{0.2614}.$$

OR

$$E(X^2) = \frac{1}{2}\lambda^{-2/2}\Gamma\left(3+\frac{2}{2}\right) = \frac{1}{2}\lambda^{-1}\cdot\Gamma(4) = \frac{3!}{2\lambda} = \frac{3}{\lambda}.$$

$$\frac{3}{\lambda} = \overline{X^2} = \frac{1}{n}\cdot\sum_{i=1}^{n} X_i^2 \qquad \Rightarrow \qquad \tilde{\lambda}_2 = \frac{3}{\overline{X^2}} = \frac{3n}{\sum_{i=1}^{n} X_i^2}.$$

$x_1 = 4, \quad x_2 = 2, \quad x_3 = 4, \quad x_4 = 3.$ $\qquad \sum_{i=1}^{n} x_i^2 = 45.$

$$\tilde{\lambda}_2 = \frac{3n}{\sum_{i=1}^{n} x_i^2} = \frac{12}{45} = \mathbf{\frac{4}{15}} \approx \mathbf{0.2667}.$$

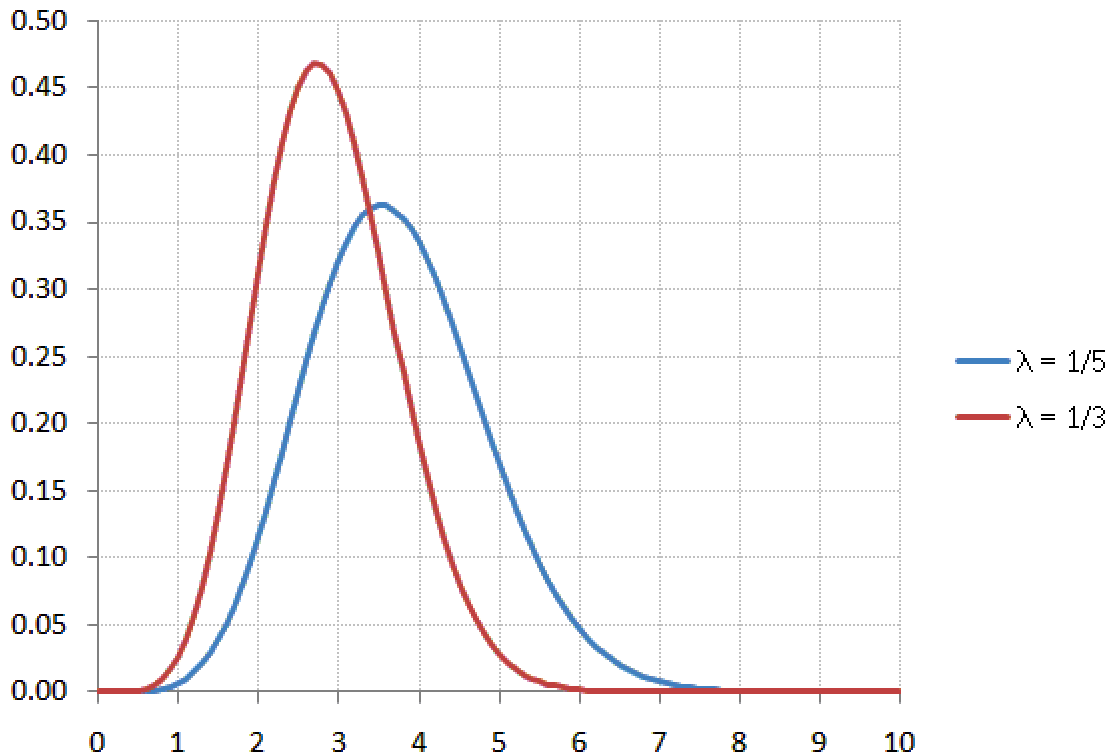
c) Obtain the maximum likelihood estimator of λ, $\hat{\lambda}$. Suppose $n = 4$, and $x_1 = 4, \quad x_2 = 2, \quad x_3 = 4, \quad x_4 = 3.$ Find the maximum likelihood estimate of λ.

$$L(\lambda) = \prod_{i=1}^{n}\left(\lambda^3 x_i^5 e^{-\lambda x_i^2}\right). \qquad \ln L(\lambda) = 3n\cdot\ln\lambda + \sum_{i=1}^{n}\ln\left(x_i^5\right) - \lambda\cdot\sum_{i=1}^{n} x_i^2.$$

$$(\ln L(\lambda))' = \frac{3n}{\lambda} - \sum_{i=1}^{n} x_i^2 = 0. \qquad \Rightarrow \qquad \hat{\lambda} = \frac{3n}{\sum_{i=1}^{n} X_i^2}.$$

$x_1 = 4, \quad x_2 = 2, \quad x_3 = 4, \quad x_4 = 3.$ $\qquad \sum_{i=1}^{n} x_i^2 = 45.$

$$\hat{\lambda} = \frac{3n}{\sum_{i=1}^{n} x_i^2} = \frac{12}{45} = \mathbf{\frac{4}{15}} \approx \mathbf{0.2667}.$$



Useful facts: **Def** $\Gamma(x) = \int_0^\infty u^{x-1} e^{-u}\, du,$ $x > 0.$

$\Gamma(1) = 1.$

$\Gamma(x) = (x-1)\,\Gamma(x-1).$

$\Rightarrow \quad \Gamma(n) = (n-1)!$ if n is an integer.

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$