STAT 400 Stepanov Dalpiaz

Examples for 8.1 (1)

0. A car manufacturer claims that, when driven at a speed of 50 miles per hour on a highway, the mileage of a certain model follows a normal distribution with mean $\mu_0 = 30$ miles per gallon and standard deviation $\sigma = 4$ miles per gallon. A consumer advocate thinks that the manufacturer is overestimating average mileage. The advocate decides to test the null hypothesis $H_0: \mu = 30$ against the alternative hypothesis $H_1: \mu < 30$.

a) Suppose the actual overall average mileage μ is indeed 30 miles per gallon. What is the probability that the sample mean is 29.4 miles per gallon or less, for a random sample of $n = 25$ cars?

$$P(\overline{X} \le 29.4) = P\left(Z \le \frac{29.4-30}{4/\sqrt{25}}\right) = P(Z \le -0.75) = \Phi(-0.75) = \mathbf{0.2266}.$$

b) A random sample of 25 cars yields $\overline{x} = 29.4$ miles per gallon. Based on the answer for part (a), is there a reason to believe that the actual overall average mileage is not 30 miles per gallon?

If $\mu = 30$, it is not unusual to see the values of the sample mean $\overline{x}$ at 29.4 miles per gallon or even lower. It does not imply that $\mu = 30$, but we have no reason to doubt the manufacturer's claim.

c) Suppose the actual overall average mileage μ is indeed 30 miles per gallon. What is the probability that the sample mean is 28 miles per gallon or less, for a random sample of $n = 25$ cars?

$$P(\overline{X} \le 28) = P\left(Z \le \frac{28-30}{4/\sqrt{25}}\right) = P(Z \le -2.50) = \Phi(-2.50) = \mathbf{0.0062}.$$

d) A random sample of 25 cars yields $\overline{x} = 28$ miles per gallon. Based on the answer for part (c), is there a reason to believe that the actual overall average mileage is not 30 miles per gallon?

If $\mu = 30$, it is very unusual to see the values of the sample mean $\overline{x}$ at 28 miles per gallon or lower. It does not imply that $\mu < 30$, but we have a very good reason to doubt the manufacturer's claim.

e) Suppose the consumer advocate tests a sample of $n = 25$ cars. What is the significance level associated with the rejection region "Reject H_0 if $\overline{x} < 28.6$"?

α = significance level = P(Type I Error) = P(Reject H_0 | H_0 true).

Need $P(\overline{X} \le 28.6 \mid \mu = 30) = ?$ $\qquad \dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} = Z.$

$$P(\overline{X} < 28.6 \mid \mu = 30) = P\left(Z < \frac{28.6 - 30}{4/\sqrt{25}} \right) = P(Z < -1.75) = \Phi(-1.75) = \mathbf{0.0401}.$$

f) Suppose the consumer advocate tests a sample of $n = 25$ cars. Find the rejection region with the significance level $\alpha = 0.05$.

$n = 25.$ $\qquad \alpha = 0.05.$

Rejection Region:

Reject H_0 if $\qquad Z = \dfrac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} < -z_\alpha. \qquad Z = \dfrac{\overline{X} - 30}{4/\sqrt{25}} < -1.645.$

$$\overline{X} < 30 - 1.645 \cdot \frac{4}{\sqrt{25}} = 28.684.$$

g) Suppose that the sample mean is $\overline{x} = 29$ miles per gallon for a sample of $n = 25$ cars. Find the p-value of the appropriate test.

$H_0: \mu \ge 30$ vs. $H_1: \mu < 30.$ $\qquad$ Left – tailed.

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{29 - 30}{4/\sqrt{25}} = \mathbf{-1.25}.$$

$$P(Z \le -1.25) = \Phi(-1.25) = \mathbf{0.1056}.$$

h) State your decision (Accept H_0 or Reject H_0) for the significance level $\alpha = 0.05$.

P-value > α $\Rightarrow$ Do NOTReject H_0 P-value < α $\Rightarrow$ Reject H_0

Since $0.1056 > 0.05$, **Do NOT Reject H_0 at $\alpha = 0.05$**.

i) Construct a 95% confidence interval for the overall average miles-per-gallon rating for this model, μ.

$\sigma = 4$ is known. $n = 25$. The confidence interval : $\overline{X} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$.

95% confidence level, $\alpha = 0.05$, $\alpha/2 = 0.025$, $z_{\alpha/2} = 1.96$.

$29 \pm 1.96 \cdot \frac{4}{\sqrt{25}}$ **29 ± 1.568** **(27.432 ; 30.568)**

j) What is the minimum sample size required if we want to estimate μ to within 0.5 miles per gallon with 95% confidence?

$\varepsilon = 0.5$, $\sigma = 4$,

95% confidence level, $\alpha = 0.05$, $\alpha/2 = 0.025$, $z_{\alpha/2} = 1.96$.

$n = \left[\frac{z_{\alpha/2} \cdot \sigma}{\varepsilon}\right]^2 = \left[\frac{1.96 \cdot 4}{0.5}\right]^2 = 245.8624.$ Round up. $n = \mathbf{246}$.

k) Construct a 95% confidence upper bound for μ.

The confidence upper bound for μ : $\overline{X} + z_{\alpha} \cdot \frac{\sigma}{\sqrt{n}}$.

95% confidence level, $\alpha = 0.05$, $z_{\alpha} = 1.645$.

$29 + 1.645 \cdot \frac{4}{\sqrt{25}}$ **29 + 1.316** **(0 ; 30.316)**

1. The overall standard deviation of the diameters of the ball bearings is $\sigma = 0.005$ mm. The overall mean diameter of the ball bearings must be 4.300 mm. A sample of 81 ball bearings had a sample mean diameter of 4.299 mm. Is there a reason to believe that the actual overall mean diameter of the ball bearings is not 4.300 mm?

a) Perform the appropriate test using a 10% level of significance.

Claim: $\mu \neq 4.300$

$H_0: \mu = 4.300$ vs. $H_1: \mu \neq 4.300$

Test Statistic: σ is known

$$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} = \frac{4.299 - 4.300}{0.005 / \sqrt{81}} = \mathbf{-1.80}.$$

Rejection Region: 2 – tailed.

Reject H_0 if $Z < -z_{\alpha/2}$ or $Z > z_{\alpha/2}$

$\alpha = 0.10$ $\alpha/2 = 0.05.$ $z_{0.05} = \mathbf{1.645}.$

Reject H_0 if $Z < -1.645$ or $Z > 1.645.$

Decision:

The value of the test statistic **does** fall into the Rejection Region.

Reject H_0 at $\alpha = 0.10$.

OR

P-value:

p-value = $P(Z \leq -1.80) + P(Z \geq 1.80) = 0.0359 + 0.0359 = \mathbf{0.0718}.$

Decision:

$0.0718 < 0.10.$ P-value $< \alpha$.

Reject H_0 at $\alpha = 0.10$.

OR

Confidence Interval:

σ is known. The confidence interval : $\overline{X} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$.

90% conf. level. $\alpha = 0.10$ $\alpha/2 = \mathbf{0.05}$. $z_{0.05} = \mathbf{1.645}$.

$4.299 \pm 1.645 \cdot \frac{0.005}{\sqrt{81}}$ **4.299 ± 0.0009138889**

Decision:

90% confidence interval for μ **does not** cover 4.300.

Reject H_0 at $\alpha = 0.10$.

Two-tailed test	same α	Confidence Interval
Accept H_0	$\Leftrightarrow$	Covers μ_0
Reject H_0	$\Leftrightarrow$	Does not cover μ_0

b) State your decision (Accept H_0 or Reject H_0) for the significance level $\alpha = 0.05$.

0.0718 > 0.05. P-value > α.

Do NOT Reject H_0 (Accept H_0) at $\alpha = 0.05$.

2. A trucking firm believes that its mean weekly loss due to damaged shipments is at most \$1800. Half a year (26 weeks) of operation shows a sample mean weekly loss of \$1921.54 with a sample standard deviation of \$249.39.

a) Perform the appropriate test. Use the significance level $\alpha = 0.10$.

Claim: $\mu \le 1800$

$H_0: \mu \le 1800$ vs. $H_1: \mu > 1800$

Test Statistic: σ is unknown

$$T = \frac{\overline{X} - \mu_0}{s / \sqrt{n}} = \frac{1921.54 - 1800}{249.39 / \sqrt{26}} = \mathbf{2.485}.$$

Rejection Region: Right – tailed.

Reject H_0 if $T > t_\alpha$

$\alpha = 0.10$. $n - 1 = 25$ degrees of freedom. $t_{0.10} = \mathbf{1.316}$.

Reject H_0 if $T > 1.316$.

Decision:

The value of the test statistic **does** fall into the Rejection Region.

Reject H_0 at $\alpha = 0.10$.

OR

P-value:

P-value = (Area to the right of $T = 2.485$) = **0.01**.

($n - 1 = 25$ degrees of freedom)

Decision:

$0.01 < 0.10$. P-value $< \alpha$.

Reject H_0 at $\alpha = 0.10$.

b) State your decision (Accept H_0 or Reject H_0) for the significance level $\alpha = 0.05$.

0.01 < 0.05. P-value < α.

Reject H_0 at $\alpha = 0.05$.

The t Distribution

r	$t_{0.40}$	$t_{0.25}$	$t_{0.20}$	$t_{0.15}$	$t_{0.10}$	$t_{0.05}$	$t_{0.025}$	$t_{0.02}$	$t_{0.01}$	$t_{0.005}$
25	0.256	0.684	0.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787

3. *Metaltech Industries* manufactures carbide drill tips used in drilling oil wells. The life of a carbide drill tip is measured by how many feet can be drilled before the tip wears out. *Metaltech* claims that under typical drilling conditions, the life of a carbide tip follows a normal distribution with mean of at least 32 feet. Suppose some customers disagree with *Metaltech*'s claims and argue that *Metaltech* is overstating the mean (i.e. the mean is actually less than 32). *Metaltech* agrees to examine a random sample of 25 carbide tips to test its claim against the customers' claim. If the *Metaltech*'s claim is rejected, *Metaltech* has agreed to give customers a price rebate on past purchases. Suppose *Metaltech* decided to use a 5% level of significance and the observed sample mean is 30.5 feet with the sample variance 16 feet^2. Perform the appropriate test.

Claim: $\mu \geq 32$ (*Metaltech*) $\mu < 32$ (customers)

$H_0: \mu \geq 32$ vs. $H_1: \mu < 32$

Test Statistic: σ is unknown

$$T = \frac{\overline{X} - \mu_0}{s/\sqrt{n}} = \frac{30.5 - 32}{4/\sqrt{25}} = \mathbf{-1.875}.$$

Rejection Region: Left – tailed.

Reject H_0 if $T < -t_{\alpha}$

$\alpha = 0.05$. $n - 1 = 24$ degrees of freedom. $t_{0.05} =$ **1.711**.

Reject H_0 if $T < -1.711$.

Decision:

The value of the test statistic **does** fall into the Rejection Region.

Reject H_0 at $\alpha = 0.05$.

OR

P-value:

Since $1.711 < 1.875 < 2.064$

$t_{0.05} < (-T) < t_{0.025}$,

$0.05 >$ P-value > 0.025.

Decision:

P-value < 0.05. P-value < α.

Reject H_0 at $\alpha = 0.05$.

The t Distribution

r	$t_{0.40}$	$t_{0.25}$	$t_{0.20}$	$t_{0.15}$	$t_{0.10}$	$t_{0.05}$	$t_{0.025}$	$t_{0.02}$	$t_{0.01}$	$t_{0.005}$
24	0.256	0.685	0.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797

=TDIST(1.875,24,1)

```
> pt(-1.875,24)
```

0.0365

4 – 5. The following random sample was obtained from $N(\mu, \sigma^2)$ distribution:

16 12 18 13 21 15 8 17

Recall: $\overline{x} = 15$, $s = 4$.

4. a) Use $\alpha = 0.05$ to test $H_0: \mu = 17$ vs. $H_1: \mu \neq 17$. Report the value of the test statistic, the critical value(s), and state your decision.

Test Statistic: $T = \dfrac{\overline{X} - \mu_0}{s / \sqrt{n}} = \dfrac{15 - 17}{4 / \sqrt{8}} \approx \mathbf{-1.414}$.

$n - 1 = 7$ degrees of freedom.

Rejection Region: $T < -t_{0.025}(7) = -2.365$ or $T > t_{0.025}(7) = 2.365$.

The value of the test statistic is **not** in the Rejection Region.

Do NOT Reject H_0 at $\alpha = 0.05$.

b) Find the p-value (approximately) of the test in part (a).

$t_{0.10}(7) = 1.415$. One tail ≈ 0.10 (a bit larger).

P-value = Two tails $\approx$ **0.20** (a bit larger).

```
=TDIST(1.414,7,2)                > 2*pt(-1.414,7)
```

0.20026

5. c) Use $\alpha = 0.05$ to test $H_0 : \mu = 12$ vs. $H_1 : \mu > 12$. Report the value of the test statistic, the critical value(s), and state your decision.

Test Statistic: $T = \dfrac{\overline{X} - \mu_0}{s / \sqrt{n}} = \dfrac{15 - 12}{4 / \sqrt{8}} \approx \mathbf{2.121}.$ (2.12132)

$n - 1 = 7$ degrees of freedom.

Rejection Region: $T > t_{0.05}(7) = 1.895.$

The value of the test statistic **is** in the Rejection Region.

Reject $\mathbf{H_0}$ at $\boldsymbol{\alpha = 0.05}$.

d) Using the *t* distribution table only, what is the p-value of the test in part (c)? (You may give a range.)

$t_{0.025}(7) = 2.365 > 2.121 > 1.895 = t_{0.05}(7).$ **0.025** < p-value < **0.05**.

e) Use a computer to find the p-value of the test in part (c).

"Hint": EXCEL =TDIST(t , degrees of freedom , 1)
gives area to the right of t.
=TDIST(t , degrees of freedom , 2)
gives $2 \times$ (area to the right of t).

OR R `> pt( t ,` degrees of freedom)
gives area to the left of t.

=TDIST(2.121,7,1) `> 1-pt(2.121,7)`

0.0358

```
> x = c(16,12,18,13,21,15,8,17)
>
> t.test(x,alternative = c("two.sided"),mu=17,conf.level=0.95)

        One Sample t-test

data:  x
t = -1.4142, df = 7, p-value = 0.2002
alternative hypothesis: true mean is not equal to 17
95 percent confidence interval:
 11.65592 18.34408
sample estimates:
mean of x
       15

> t.test(x,alternative = c("greater"),mu=12,conf.level=0.95)

        One Sample t-test

data:  x
t = 2.1213, df = 7, p-value = 0.03579
alternative hypothesis: true mean is greater than 12
95 percent confidence interval:
 12.32066      Inf
sample estimates:
mean of x
       15
```